UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 5 June, 2014 1:30 pm to 4:30 pm

PAPER 27

STOCHASTIC CALCULUS AND APPLICATIONS

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let X be a continuous local martingale with $X_0 = 0$, such that $\mathbb{E}(\langle X\rangle_t^{p/2}) < \infty$ for all $t \geqslant 0$ and $p \geqslant 2$.

(a) By applying Itô's formula to $|X_t|^p$ or otherwise, show that if X is uniformly bounded then

$$\mathbb{E}(|X_t|^p) \leqslant \frac{p(p-1)}{2}\mathbb{E}\left(\sup_{0\leqslant s\leqslant t}|X_s|^{p-2}\langle X\rangle_t\right)$$

for all $p \geqslant 2$. Conclude that there is a constant C_p, depending only on p, such that

$$\mathbb{E}\left(\sup_{0\leqslant s\leqslant t}|X_s|^p\right) \leqslant C_p\mathbb{E}(\langle X\rangle_t^{p/2}). \qquad (*)$$

Show that inequality $(*)$ remains valid even if X is unbounded. [You may use without proof Doob's inequality: for any continuous martingale M we have

$$\mathbb{E}\left(\sup_{0\leqslant s\leqslant t}|M_s|^p\right) \leqslant \frac{p^p}{(p-1)^p}\mathbb{E}(|M_t|^p)$$

for all $p > 1$.]

(b) Show that $Y_t = X_t^4 - 6X_t^2\langle X\rangle_t + 3\langle X\rangle_t^2$ defines a martingale. Assuming that

$$\mathbb{E}(X_t^2) = t, \quad \mathrm{Cov}(X_t^2, \langle X\rangle_t) = 0 \quad \text{for all } t \geqslant 0,$$

show that $\mathbb{E}(X_t^4) \leqslant 3t^2$; furthermore, show that if $\mathbb{E}(X_t^4) = 3t^2$ for all $t \geqslant 0$, then X is a Brownian motion.

2

(a) Let $(M_t)_{t \geqslant 0}$ be a continuous local martingale with $M_0 = 0$ with respect to a filtration $(\mathcal{F}_t)_{t \geqslant 0}$ satisfying the usual conditions. State and prove the Dambis–Dubins–Schwarz theorem in terms of M under the extra assumption that $\langle M \rangle$ is strictly increasing.

(b) Let X and Y be independent Brownian motions, and let

$$R_t = \sqrt{X_t^2 + Y_t^2}.$$

Show that there exists a Brownian motion W and an increasing adapted process A such that

$$R_t^2 = 2W_{A(t)} + 2t.$$

Now let

$$Z_t = \int_0^t Y_s dX_s - X_s dY_s.$$

Show that there is a Brownian motion B which is independent of W and such that

$$Z_t = B_{A(t)}.$$

3

Let $g_n(t) = \sqrt{2}\cos[(n - \frac{1}{2})\pi t]$ and $h_n(t) = \sqrt{2}\sin[(n - \frac{1}{2})\pi t]$, and let W be a Brownian motion. In this question you may use without proof the fact that the collections $(g_n)_{n \geqslant 1}$ and $(h_n)_{n \geqslant 1}$ are both orthonormal bases of $L^2[0,1]$.

(a) Let $\xi_n = \int_0^1 g_n(t) dW_t$. Show that the sequence $(\xi_n)_{n \geqslant 1}$ are independent $N(0,1)$ random variables.

(b) Show that $\xi_n = (n - \frac{1}{2})\pi \int_0^1 h_n(t) W_t dt$ and hence conclude that

$$\int_0^1 W_t^2 dt = \sum_{n=1}^{\infty} \frac{1}{(n - \frac{1}{2})^2 \pi^2} \xi_n^2.$$

(c) Compute the Laplace transform

$$\mathbb{E}(e^{-\lambda \int_0^1 W_t^2 dt})$$

in terms of $\lambda \geqslant 0$. [You might find it useful to note that

$$\prod_{n=1}^{\infty} \left(1 + \frac{x^2}{(n - \frac{1}{2})^2 \pi^2} \right) = \cosh x$$

for all $x \in \mathbb{R}$.]

4

Let Z be a positive continuous uniformly integrable martingale with $Z_0 = 1$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define an equivalent measure $\mathbb{Q} \sim \mathbb{P}$ on $(\Omega, \mathcal{F})$ by the density

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_\infty.$$

Let X be a continuous $\mathbb{P}$-local martingale with $X_0 = 0$, and let

$$Y_t = X_t - \langle \log Z, X \rangle_t.$$

Show that the process Y is a $\mathbb{Q}$-local martingale. [If you use Girsanov's theorem, you must prove it.]

Now suppose W is Brownian motion defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and suppose that W generates the filtration. Show that there exists a predictable process α such that

$$\int_0^t \alpha_s^2 ds < \infty \text{ almost surely for all } t \geqslant 0$$

and such that the process

$$\hat{W}_t = W_t - \int_0^t \alpha_s ds$$

is a $\mathbb{Q}$-Brownian motion. [You may appeal to any standard integral representation results if clearly stated.]

5

Let M be a continuous, non-negative local martingale such that $M_0 = 1$ and $M_t \to 0$ almost surely as $t \to \infty$.

(a) If M is strictly positive, show that $M_t = e^{X_t - \langle X \rangle_t / 2}$ for a continuous local martingale X such that $\langle X \rangle_\infty = \infty$ almost surely.

(b) For each $a > 1$, let $T_a = \inf\{t \geqslant 0 : M_t > a\}$. Show that

$$\mathbb{P}(T_a < \infty) = \mathbb{P}(\sup_{t \geqslant 0} M_t > a) = 1/a.$$

[*Hint: Compute the expected value of* $M_{t \wedge T_a} = a\mathbf{1}_{\{T_a \leqslant t\}} + M_t \mathbf{1}_{\{T_a > t\}}$.]

(c) Let W be a Brownian motion. Find the density functions of the following random variables.

1. $\sup_{0 \leqslant t \leqslant \tau(-b)} W_t$ where $\tau(-b) = \inf\{t \geqslant 0 : W_t < -b\}$ and $b > 0$.
2. $\sup_{t \geqslant 0}(W_t - \lambda t)$ for $\lambda > 0$.

6

Consider the stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \qquad (*)$$

where W is a Brownian motion and the functions b and σ are bounded and smooth. Assume that for every square-integrable ξ independent of W, there exists a unique strong solution X such that $X_0 = \xi$ and $\sup_{t \geqslant 0} \mathbb{E}(X_t^2) < \infty$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function and let $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a bounded and smooth solution of the PDE

$$\frac{\partial u}{\partial t} = b(x)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma(x)^2 \frac{\partial^2 u}{\partial x^2},$$

with boundary condition

$$u(0, x) = f(x) \text{ for all } x \in \mathbb{R}.$$

(a) Show that $u(t, x) = \mathbb{E}\left[f(X_t) | X_0 = x\right]$

Suppose b/σ^2 is locally integrable and let

$$p(x) = \frac{C}{\sigma(x)^2} \exp\left(\int_0^x \frac{2b(s)}{\sigma(s)^2} ds\right)$$

where $C > 0$ is chosen so that $\int_{-\infty}^{\infty} p(x)dx = 1$. Assume $\int_{-\infty}^{\infty} x^2 p(x)dx < \infty$.

(b) Briefly show that

$$\int u(t, x)p(x)dx = \int_{-\infty}^{\infty} f(x)p(x)dx$$

for all $t \geqslant 0$. You may integrate by parts and apply Fubini's theorem without justification.

Now suppose there is a constant $k > 0$ such that

$$2(x - y)[b(x) - b(y)] + [\sigma(x) - \sigma(y)]^2 \leqslant -k(x - y)^2.$$

(c) Let Y be another strong solution of $(*)$. Show that

$$\mathbb{E}[(X_t - Y_t)^2] \leqslant \mathbb{E}[(X_0 - Y_0)^2]e^{-kt}.$$

[*Hint: You may use this version of Gronwall's lemma: If h is locally integrable and*

$$h(t) \leqslant h(s) - k\int_s^t h(u)du \text{ for all } 0 \leqslant s \leqslant t,$$

then $h(t) \leqslant h(0)e^{-kt}$ for all $t \geqslant 0$.]

(d) Show that

$$u(t, x) \to \int_{-\infty}^{\infty} f(y)p(y)dy \text{ as } t \to \infty.$$

for all $x \in \mathbb{R}$.

END OF PAPER