UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 29 May, 2014 1:30 pm to 4:30 pm

PAPER 26

ADVANCED PROBABILITY

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $X_1, X_2, \ldots$ be i.i.d. integrable random variables in $\mathbb{R}$ with $\mathbb{E}[X_i] = 0$ and $\mathbb{P}(X_i > 0) > 0$. Let $x > 0$, $S_0 = x$, and $S_n = x + \sum_{i=1}^{n} X_i$. For every $0 < r < \infty$ we define

$$\eta = \inf\{n \geqslant 0 : S_n \leqslant 0 \quad \text{or} \quad S_n \geqslant r\}.$$

1) Show that $\mathbb{E}[\eta] < \infty$.
[*Hint: Note that the condition* $\mathbb{P}(X_j > 0) > 0$ *implies the existence of* $\delta > 0$ *and* $m \in \mathbb{N}$ *such that* $\mathbb{P}(X_j > r/m) > \delta$.]

2) Show that X_η is integrable.

3) Show that $\mathbb{E}S_\eta = x$.

2

1) Let X be an integrable random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra. State the definition of the conditional expectation of X given $\mathcal{G}$.

2) Prove the following version of the Optional Stopping Theorem: Let M be a discrete-time martingale and let T be a bounded stopping time. Then $\mathbb{E}[M_T] = \mathbb{E}[M_0]$.

3) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G}$ be a sub σ-algebra. Suppose that X, Y are bounded random variables satisfying

$$\mathbb{E}[Y|\mathcal{G}] = X \quad \text{a.s.} \quad \text{and} \quad \mathbb{E}[X^2] = \mathbb{E}[Y^2].$$

Show that $X = Y$ almost surely.

4) Let $(X_n)_n$ be a martingale on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n), \mathbb{P})$ and let T be a stopping time satisfying

$$\mathbb{P}(T < \infty) = 1, \quad \mathbb{E}[|X_T|] < \infty \quad \text{and} \quad \mathbb{E}[|X_n|\mathbf{1}(T > n)] \to 0 \,\text{as}\, n \to \infty.$$

Show that $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

3

Let B be a standard Brownian motion in $\mathbb{R}$ and let $(\mathcal{F}_t)$ be its natural filtration.

1) Let $a \neq 0$. Show that $(B_t + at)_t$ is transient, in the sense that, for all $x \in \mathbb{R}$, almost surely the set of times

$$\{t \geqslant 0 : B_t + at = x\}$$

is bounded.

2) Let $M_t = \sup_{0 \leqslant s \leqslant t} B_s$. Find the joint distribution function of (B_t, M_t), i.e. probabilities of the form $\mathbb{P}(M_t \geqslant a, B_t \leqslant b)$ for all $a, b \in \mathbb{R}$.

3) Consider the random time

$$\tau = \inf \left\{ t \geqslant 0 : B_t = \max_{0 \leqslant s \leqslant 1} B_s \right\}.$$

i) Show that $\tau < 1$ almost surely.

ii) Is the process $(B_{t+\tau} - B_\tau)_{t \geqslant 0}$ a Brownian motion?

iii) Is τ an $(\mathcal{F}_t)$-stopping time? Justify your answer.

4

1) Show that Brownian motion is locally α-Hölder continuous for any $\alpha < 1/2$ almost surely. [State carefully any results you appeal to.]

2) Let B be a standard Brownian motion in $\mathbb{R}$. Show that, almost surely, B is not differentiable at 0.

3) Let B be a standard Brownian motion in $\mathbb{R}$. Show that almost surely for all $0 < a < b < \infty$, the process B is not monotone on the time interval $[a, b]$.

5

Suppose $X_1, X_2, \ldots$ are i.i.d. random variables with $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_1^2 = 1$. Let $S_n = X_1 + \ldots + X_n$ be the associated random walk with $S_0 = 0$, and let

$$M_n = \max\{S_k, 0 \leqslant k \leqslant n\}$$

be its maximal value up to time n. Show that for all $x \geqslant 0$

$$\lim_{n \to \infty} \mathbb{P}(M_n \geqslant x\sqrt{n}) = \frac{2}{\sqrt{2\pi}} \int_x^\infty e^{-y^2/2}\, dy.$$

[State carefully any results from the course you appeal to without including their proofs.]

6

a) State the definition of a Lévy process and a Poisson random measure.

b) Prove that the sum of two independent Lévy processes is again a Lévy process.

c) Let a, b, and c be non-negative real parameters, and let $X = (X_t \colon t \geqslant 0)$ be a Lévy process with characteristic function given by

$$\phi_{X_t}(u) = \exp\left[t\left(a\cos(bu) + ce^{iu} - a - c(1 - iu)\right)\right].$$

Find a representation of X in terms of a Poisson random measure, and describe the sample paths of the process X.

END OF PAPER