UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 6 June, 2014 9:00 am to 12:00 pm

PAPER 24

ALGEBRAIC NUMBER THEORY

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

What is an *absolute value* on a field K? What does it mean to say that an absolute value is *nonarchimedean*? What does it mean to say that two absolute values are *equivalent*? Show that there is a bijection between the set of equivalence classes of nonarchimedean absolute values on K and the set of equivalence classes of valuations on K.

Show that any nonarchimedean absolute value of $\mathbb{Q}$ is equivalent to some p-adic absolute value.

Let K be complete with respect to a nonarchimedean absolute value. What is the valuation ring R of K? Show that R is compact (with respect to the valuation topology) if and only if (i) K is discretely valued, and (ii) the residue field of K is finite.

2

Let L/K be a finite separable extension of fields complete with respect to a discrete valuation. Define the *inverse different* of L/K, and show that it is the inverse of an ideal $\mathcal{D}_{L/K}$ of $\mathfrak{o}_L$.

Suppose that $\mathfrak{o}_L = \mathfrak{o}_K[x]$, and that g is the minimal polynomial of x over K. Show that $\mathcal{D}_{L/K} = g'(x)\mathfrak{o}_L$. Deduce that if L/K is totally ramified of degree n with residue characteristic p, where $(p,n) = 1$, then $\mathcal{D}_{L/K} = \pi_L^{n-1}\mathfrak{o}_L$.

Compute, for any positive integer r, the different of the extension $K_r = \mathbb{Q}_p(\zeta_{p^r})/\mathbb{Q}_p$, where ζ_{p^r} is a primitive p^r-th root of unity.

[You may assume without proof that for a finite separable extension of fields, the trace form is non-degenerate.]

3

Let p be an odd prime and $K = \mathbb{Q}_p(\zeta_p)$, $\pi_K = 1 - \zeta_p$.

(i) Show that if $1 \leqslant i \leqslant p-1$ then $(1-\zeta_p^i)/(1-\zeta_p) \equiv i \pmod{\pi_K}$. Deduce that $(1-\zeta_p)^{p-1} = -pu$ for some $u \in 1 + \pi_K\mathfrak{o}_K$.

(ii) Use Hensel's Lemma to show that for every $u \in 1 + \pi_K\mathfrak{o}_K$ there exists $v \in \mathfrak{o}_K^*$ with $v^{p-1} = u$, and deduce that $K = \mathbb{Q}_p(\sqrt[p-1]{-p})$.

(iii) Let $L = \mathbb{Q}_p(\sqrt[p(p-1)]{-p})$. Deduce that $L/\mathbb{Q}_p$ is the splitting field of $X^p - p$. Find the ramification groups of $L/\mathbb{Q}_p$.

4

Define the group of *ideles* J_K of a number field K. What is the topology on J_K? Show that K^* is a discrete subgroup of J_K. Let

$$U_K = \prod_{v \text{ infinite}} K_v^* \times \prod_{v \text{ finite}} \mathfrak{O}_v^*.$$

Show that J_K/K^*U_K is isomorphic to the class group of K.

Assuming the compactness of J_K^1/K^*, show that the class group of K is finite, and give an outline proof of Dirichlet's Unit Theorem.

5

Let $F/\mathbb{Q}_p$ be a finite extension, with valuation ring $\mathfrak{o}_F$ and uniformiser π_F. What is the *Schwartz space* $\mathcal{S}(F)$? For $f \in \mathcal{S}(F)$, explain carefully what is meant by the Fourier transform of f.

Let $a \in F$ and $n \in \mathbb{Z}$. Compute the Fourier transform of the characteristic function of $a+\pi_F^n\mathfrak{o}_F$. Deduce that for every $f \in \mathcal{S}(F)$, its Fourier transform $\widehat{f}$ also belongs to $\mathcal{S}(F)$, and that (for appropriate choices of Haar measure and additive character) $\widehat{\widehat{f}}(x) = f(-x)$.

END OF PAPER