UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Thursday, 5 June, 2014 1:30 pm to 4:30 pm

## PAPER 23

## MODULAR FORMS

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

a) Briefly explain the construction of the Riemann surface $X(\Gamma(1))$ as a quotient of the extended upper half-plane $\mathcal{H} \coprod \mathbb{P}^1(\mathbb{Q})$. Describe the points where the map $\mathcal{H} \to X(\Gamma(1))$ is ramified and the ramification index at these points. Prove that $X(\Gamma(1))$ is compact.

Let $\Delta$ be the unique normalised cusp form in $S_{12}(\Gamma(1))$ and let $E_4$ be the normalised weight 4 Eisenstein series

$$E_4 = 1 + 240 \sum_{n \geqslant 1} \sigma_3(n) q^n.$$

b) Show that the function $j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)}$ induces a biholomorphism from $X(\Gamma(1))$ to $\mathbb{P}^1_{\mathbb{C}}$, and that $j(\tau)$ has a zero at $\omega = \frac{-1+\sqrt{3}i}{2}$. [*You may assume that $\Delta(\tau)$ is non-vanishing on $\mathcal{H}$ and has $q$-expansion $q + \cdots$.*]

c) Show that the function $j_2(\tau) = \frac{\Delta(\tau)}{\Delta(2\tau)}$ induces a biholomorphism from $X(\Gamma_0(2))$ to $\mathbb{P}^1_{\mathbb{C}}$.

d) Show that

$$j(\tau) = \frac{(j_2(\tau) - j_2(\omega))^3}{j_2(\tau)^2}.$$

**2**

For $\tau \in \mathcal{H}$ and $q = e^{2\pi i\tau}$ the Jacobi $\theta$-function is defined by

$$\theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}.$$

Define another function $\phi$ on $\mathcal{H}$ by

$$\phi(\tau) = \theta\left(\tau - \frac{1}{2}\right) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}.$$

a) Show that, for $\tau \in \mathcal{H}$,

$$\theta\left(\frac{-1}{4\tau}\right) = \sqrt{\frac{2\tau}{i}}\theta(\tau).$$

Hence (or otherwise) show that, for positive integers $k$, $\phi^{8k}$ is an element of $M_{4k}(\Gamma_0(2))$.

[*You may quote the Poisson summation formula without proof, and assume that* $\Gamma_0(2)$ *is generated by the matrices*

$$\pm\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \pm\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}.]$$

b) Show that $S_4(\Gamma_0(2)) = \{0\}$ and $\dim M_4(\Gamma_0(2)) = 2$.

[*Hint: for* $f$ *in* $S_4(\Gamma_0(2))$ *consider the product*

$$\prod_{\alpha\in\Gamma_0(2)\backslash \mathrm{SL}_2(\mathbb{Z})} f|_{\alpha,4}.]$$

c) Show that

$$\phi^8 = \sum_{n\geqslant 0} (-1)^n r_8(n) q^n$$

where the integers $r_8(n)$ satisfy

$$r_8(n) = (-1)^n 16 \sum_{0<d|n} (-1)^d d^3.$$

[*It may be helpful to recall that the normalised weight* 4 *Eisenstein series has q-expansion*

$$E_4 = 1 + 240\sum_{n\geqslant 1}\sigma_3(n)q^n.]$$

**3**

a) Let $\Gamma$ be a congruence subgroup. Show that the genus of the modular curve $X(\Gamma)$ is given by

$$1 + \frac{d}{12} - \frac{r_2}{4} - \frac{r_3}{3} - \frac{r_\infty}{2}$$

where $d$ is the index of the image $\overline{\Gamma}$ of $\Gamma$ in $\mathrm{PSL}_2(\mathbb{Z})$, $r_\infty$ is the number of cusps of $X(\Gamma)$ and, for $i = 2, 3$, $r_i$ is the number of $\Gamma$-equivalence classes of points in $\mathcal{H}$ whose stabiliser in $\overline{\Gamma}$ has order $i$.

Now let $p$ be a prime number with $p \geqslant 5$.

b) Show that the modular curve $X(\Gamma_1(p))$ has genus

$$\frac{(p-5)(p-7)}{24}.$$

c) Suppose $k$ is a positive integer and $f$ is a non-zero meromorphic form of weight $k$ and level $\Gamma_1(p)$. Write down a divisor $D(f)$ on $X(\Gamma_1(p))$ such that

$$S_k(\Gamma_1(p)) = \{f\phi : \phi \in \mathcal{L}(D(f))\},$$

justifying this equality.

Assuming the existence of such an $f$, show that for $k \geqslant 3$

$$\dim S_k(\Gamma_1(p)) = \frac{(p-1)}{24}((p+1)(k-1) - 12).$$

[*You may assume any general results about Riemann surfaces without proof, so long as they are clearly stated.*]

**4**

Let $N$ be a positive integer and $\chi$ a homomorphism

$$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times.$$

Denote by $\mathcal{L}_N$ the set of pairs $(L, t)$ where $L \subset \mathbb{C}$ is a lattice and $t$ is an element of $\mathbb{C}/L$ of order $N$.

a) Explain how to associate a function on the set $\mathcal{L}_N$ to an element of $M_k(\Gamma_1(N))$.

b) For $p$ prime and $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, define the Hecke operators $T_p$ and $\langle d \rangle$ on the space of functions on $\mathcal{L}_N$. Show that these operators preserve the space of functions arising from elements of $M_k(\Gamma_1(N))$.

Denote by $M_k(N, \chi)$ the subspace of $M_k(\Gamma_1(N))$ consisting of $f$ satisfying $\langle d \rangle f = \chi(d) f$ for all $d \in (\mathbb{Z}/N\mathbb{Z})^\times$.

c) Suppose $f = \sum_{n \geqslant 0} a_n q^n \in M_k(N, \chi)$. Show that the $q$-expansion of $T_p f$ is given by

$$T_p f = \sum_{n \geqslant 0} b_n q^n$$

where $b_n = a_{pn}$ if $p \nmid n$ and $b_n = a_{pn} + \chi(p) p^{k-1} a_{n/p}$ if $p \mid n$.

d) Suppose $f \in M_k(N, \chi)$ with $T_p(f) = \lambda f$ for some prime $p$ with $p \nmid N$. Moreover, suppose that the polynomial $X^2 - \lambda X + \chi(p) p^{k-1}$ has two distinct roots $\alpha$ and $\beta$. Denote by $\tilde{\chi}$ the homomorphism

$$\tilde{\chi} : (\mathbb{Z}/Np\mathbb{Z})^\times \to \mathbb{C}^\times$$

given by composing $\chi$ with the reduction mod $N$ map from $(\mathbb{Z}/Np\mathbb{Z})^\times$ to $(\mathbb{Z}/N\mathbb{Z})^\times$.

Show that the functions $f(\tau)$ and $f(p\tau)$ span a two-dimensional subspace of $M_k(Np, \tilde{\chi})$ on which $T_p$ acts with eigenvalues $\alpha$ and $\beta$.

[You may assume without proof that $f(p\tau)$ is holomorphic at the cusps.]

**5**

Suppose $k$ and $N$ are positive integers and let

$$f = \sum_{n \geqslant 1} a_n q^n \in S_k(\Gamma_1(N)).$$

Consider the $L$-function

$$L(f,s) = \sum_{n \geqslant 1} \frac{a_n}{n^s}.$$

a) Show that this series defines a holomorphic function for $\mathrm{Re}(s) > k/2 + 1$.

Set $\Lambda(f,s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(f,s)$ and $g(\tau) = i^k N^{-k/2}\tau^{-k} f(-1/N\tau)$.

b) Show that $\Lambda(f,s)$ extends to a holomorphic function for all $s \in \mathbb{C}$ and $\Lambda(f,s) = \Lambda(g, k-s)$.

c) Consider the function defined by $f(\tau) = q\prod_{n \geqslant 1}(1-q^n)^2(1-q^{11n})^2$, which you may assume is an element of the one-dimensional space $S_2(\Gamma_0(11))$. Show that

$$-\frac{1}{11}\tau^{-2} f(-1/11\tau) = f(\tau).$$

Deduce that the order of vanishing of $\Lambda(f,s)$ at $s = 1$ is even.

## END OF PAPER