UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 30 May, 2014 9:00 am to 12:00 pm

PAPER 20

TOPOS THEORY

*Attempt no more than **THREE** questions, of which at most **TWO** should be from Section A.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

(i) Show that if $\mathbb{G}$ is a comonad on a topos $\mathcal{E}$, whose functor part preserves finite limits, then the category of $\mathbb{G}$-coalgebras is a topos.

(ii) Recall that an object A of a topos is called *decidable* if the diagonal $A \rightarrowtail A \times A$ is a complemented subobject. Verify that decidability is inherited by arbitrary subobjects and coproducts, and by finite products.

(iii) Let $\mathcal{E}$ be a Grothendieck topos, and let $\mathcal{E}_{qd}$ denote the full subcategory of quotients of decidable objects in $\mathcal{E}$ (i.e. those B for which there exists an epimorphism $A \twoheadrightarrow B$ with A decidable). Show that $\mathcal{E}_{qd}$ is a topos.

2

Explain what is meant by a *local operator* on a topos $\mathcal{E}$, and by a *sheaf* for a local operator. Sketch the proof that if j is a local operator on $\mathcal{E}$ then the category $\mathbf{sh}_j(\mathcal{E})$ of j-sheaves is a topos, and that it is reflective in $\mathcal{E}$.

Define the *open* and *closed* local operators $o(U)$, $c(U)$ associated with a subterminal object $U \rightarrowtail 1$ in $\mathcal{E}$. Show that the $c(U)$-closed monomorphisms coincide with the $o(U)$-dense ones, and deduce that $o(U)$ and $c(U)$ are complementary elements of the lattice $\mathbf{Lop}(\mathcal{E})$ of local operators on $\mathcal{E}$. [*Hint: Which monomorphisms are both $o(U)$-dense and $c(U)$-dense? And which can be written as a composite of $c(U)$-dense and $o(U)$-dense monomorphisms?*]

3

Explain what is meant by a *coherent theory* over a first-order signature Σ. List the axioms and rules of inference of coherent logic. Define the *syntactic category* $\mathcal{C}_{\mathbb{T}}$ of a coherent theory $\mathbb{T}$, and sketch the proof that it is a coherent category and contains a conservative model of $\mathbb{T}$. [Detailed verifications of derivability in $\mathbb{T}$ are not required.]

SECTION B

4

(i) Explain briefly how a functor $F : \mathcal{C} \to \mathcal{D}$ between small categories induces a geometric morphism $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]$.

(ii) An object A of a cocomplete category is called an *indecomposable projective* if, whenever we have an epimorphism $\coprod_{i \in I} B_i \twoheadrightarrow A$, there exists i such that $B_i \to A$ is split epic. Show that representable functors are indecomposable projectives in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, and that the converse holds if idempotents split in $\mathcal{C}$.

(iii) Let $\mathcal{C}$ and $\mathcal{D}$ be small categories such that idempotents split in $\mathcal{D}$, and let $f \colon [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]$ be a geometric morphism such that f^* has a left adjoint $f_!$, as well as its right adjoint f_*. Show that f is induced as in (i) by a functor $\mathcal{C} \to \mathcal{D}$. [*Hint: Consider the restriction of $f_!$ to representables.*]

(iv) A topos $\mathcal{E}$ admitting a geometric morphism to $\mathbf{Set}$ is said to be *local* if the direct image functor $\mathcal{E} \to \mathbf{Set}$ is also an inverse image functor. Under what conditions on $\mathcal{C}$ is $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ local? [You may assume that idempotents split in $\mathcal{C}$; recall that $\mathbf{Set} \cong [\mathbf{1}, \mathbf{Set}]$ where $\mathbf{1}$ is the terminal category.]

5

Let $\mathcal{C}$ be a small category. Give, with justification, a necessary and sufficient condition on $\mathcal{C}$ for the assignment

$$J(U) = \{\text{all nonempty sieves on } U\}$$

to define a Grothendieck coverage on $\mathcal{C}$. Show that this condition fails if $\mathcal{C}$ is the category of nonempty finite sets and all functions between them, but that it holds for the category $\mathcal{D}$ of nonempty finite sets and surjections. Show also that every representable functor $\mathcal{D}^{\mathrm{op}} \to \mathbf{Set}$ is a sheaf for this coverage. [*Hint: Every morphism of $\mathcal{D}$ is regular epic.*]

Now let F be any J-sheaf on $\mathcal{D}$. We define an element $x \in F(n)$ to be *primitive* if it is not in the image of $F(\alpha)$ for any surjection $\alpha \colon n \to (n-1)$. [In particular, all elements of $F(1)$ are primitive.] Suppose that we have primitive elements $x \in F(m)$, $y \in F(n)$ and surjections $\alpha \colon P \to m$, $\beta \colon p \to n$ such that $F(\alpha)(x) = F(\beta)(y)$; show that two elements of p have the same image under α iff they do so under β, and deduce that there is a bijection $\gamma \colon m \to n$ with $F(\gamma)(y) = x$. [Primitive elements related in this way are said to be *equivalent*.]

By considering the subfunctors of F generated by (equivalence classes of) primitive elements, show that F may be written as a coproduct of sheaves which are epimorphic images of representables. Deduce that the subobject classifier of $\mathbf{Sh}(\mathcal{D}, J)$ is the constant functor with value $\{\bot, \top\}$.

6

(i) Let $\mathbb{T}$ be a (finitary) algebraic theory. Explain, without detailed proof, what is meant by the statement that the functor category $[\mathbb{T}_{fp}, \mathbf{Set}]$ is a *classifying topos* for $\mathbb{T}$-models in Grothendieck toposes, where $\mathbb{T}_{fp}$ may be viewed either as the category of finitely-presented $\mathbb{T}$-models in **Set**, or as an appropriate full subcategory of the opposite of the syntactic category of $\mathbb{T}$. What is the generic $\mathbb{T}$-model in this topos?

(ii) The theory of integral domains is obtained from the theory of (commutative, unitary) rings by adding the axioms $((0 = 1) \vdash \bot)$ and

$$((xy = 0) \vdash_{x,y} ((x = 0) \vee (y = 0))) \ .$$

Explain how a classifying topos for integral domains may be obtained by imposing a suitable Grothendieck coverage on the opposite of the category of finitely-presented rings. Is this coverage standard? [You need not identify all the covers explicitly.]

(iii) Show that the generic integral domain satisfies the non-coherent sequent

$$(\neg(\bigwedge_{i=1}^{n}(\exists y_i)(x_i y_i = 1)) \vdash_{x_1,\ldots,x_n} \bigvee_{i=1}^{n}(x_i = 0))$$

for all $n \geqslant 1$. Show also that a nontrivial ring (in any topos) satisfying the case $n = 2$ of this sequent is an integral domain.

END OF PAPER