UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 2 June, 2014 9:00 am to 12:00 pm

PAPER 2

LIE ALGEBRAS AND THEIR REPRESENTATIONS

Attempt no more than ***THREE*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
Triangular graph paper

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

What does it mean for a Lie algebra to be (i) abelian; (ii) solvable; (iii) nilpotent? For each pair of the above conditions indicate whether one implies the other. (i.e. Does abelian imply solvable? Does solvable imply abelian? etc.)

State and prove Lie's theorem.

From Lie's theorem, deduce the following: suppose $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ with $\mathfrak{g}$ nilpotent. Then there is a basis of V such that the elements of $\mathfrak{g}$ are represented by upper triangular matrices.

Explain why this latter result does not immediately imply Engel's theorem.

2

State and prove Weyl's theorem on the complete reducibility of $\mathfrak{g}$-modules.

[You may assume the following: Schur's lemma; the existence and any properties of the Casimir operator; any result on one-dimensional representations provided it is stated carefully.]

Give an example where, after weakening the hypotheses of Weyl's theorem, a simple Lie algebra does not act completely reducibly. You do not need to prove your example has the desired property.

3

Let $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$. On triangular graph paper, draw out the irreducible representations $\Gamma_{2,1}$ and $\Gamma_{1,0}$. What is the dimension of the representation $V := \Gamma_{2,1} \otimes S^2\Gamma_{1,0}$? Using more graph paper as needed, calculate the decomposition of V irreducible submodules.

[*Hint: Calculate carefully the multiplicities in V of the dominant weights* $(4,1)$, $(2,2)$, $(3,0)$, $(0,3)$, $(1,1)$. *Check your answer is correct using the dimension formula* $\dim \Gamma_{a,b} = (a+1)(b+1)(a+b+2)/2$.]

4

State the axioms defining an abstract root system R of a Lie algebra.

If $(\ ,\ )$ is the inner product on R and α and β are roots in R, prove that the integer $n_{\alpha\beta} = 2(\alpha,\beta)/(\beta,\beta)$ satisfies $0 \leqslant |n_{\alpha\beta}| \leqslant 4$.

What is meant by a base of simple roots for R? Assuming a base of simple roots exists, prove that if $\alpha \neq \beta$ are simple roots, then $n_{\alpha\beta}$ is negative. Thus deduce each $n_{\alpha\beta}$ with α and β distinct simple roots is in the set $\{-3,-2,-1,0\}$. [Note that zero is considered to be a negative number.]

Explain how to produce a Dynkin diagram from a root system.

List (without proof) all the connected Dynkin diagrams.

Prove that a cyclic graph is not a Dynkin diagram.

5

Define the Killing form B_V of a Lie algebra $\mathfrak{g}$ relative to a representation V of $\mathfrak{g}$.

Prove that B_V is a symmetric bilinear form and that $\mathfrak{g}$ preserves its own Killing form, i.e. $B_V([X,Y],Z) = B_V(X,[Y,Z])$.

State and prove Cartan's criterion. [Any elementary facts about the Jordan decomposition may be used without proof. You may assume Engel's theorem if it is stated carefully.]

Define what it means for a Lie algebra to be semisimple. Deduce from Cartan's criterion that a finite-dimensional complex Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form B is non-degenerate.

6

Define a (real) Lie group.

Define the Lie algebra of a Lie group by defining the tangent space at the identity and a Lie bracket on the tangent space. [You do not have to prove that the bracket is antisymmetric or that it satisfies the Jacobi identity.]

Show, by differentiating the determinant, that Lie $\mathrm{SL}_n = \mathfrak{sl}_n$.

Define the exponential and logarithm maps on GL_n. Give a brief reason as to why the exponential map is surjective on $\mathrm{GL}_n(\mathbb{C})$. Give an example of a connected Lie group where the exponential map is not surjective.

END OF PAPER