UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 29 May, 2014 1:30 pm to 4:30 pm

PAPER 19

TOPICS IN SET THEORY

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Define carefully any <u>two</u> of the following three italicized concepts:

(a) the formula $\varphi(x_0, \ldots, x_{n-1})$ is a Δ_1-*formula* in the language of ZFC;

(b) the set $X \subseteq \mathbb{R}$ is a *Sierpinski set*;

(c) the *Reflection Principle* for the cumulative hierarchy $\{V_\alpha : \alpha \in Ord\}$.

(ii) Suppose κ is an infinite cardinal. For a set X, let $[X]^{<\kappa} = \{Y : Y \subseteq X, \mid Y \mid < \kappa\}$. Let $A_{<\kappa}(X)$ be the following assertion:

for every function $f : X \to [X]^{<\kappa}$ there exist x_1 and x_2 such that

$$x_1 \notin f(x_2) \text{ and } x_2 \notin f(x_1).$$

(a) Prove (in ZFC) that $2^{\aleph_0} > \aleph_1$ implies $A_{<\aleph_1}(\mathbb{R})$.

(b) Show the converse is also true: $A_{<\aleph_1}(\mathbb{R})$ implies $2^{\aleph_0} > \aleph_1$.

(c) Deduce that $A_{<\aleph_1}(\mathbb{R})$ is independent of ZFC, stating carefully any independence results that you require.

(iii) Let ZFC^2 denote the second-order axiomatic system obtained from ZFC in which the Replacement Schema is expressed as the following single axiom with a universal second-order quantifier: $(\forall C)(\forall z \forall u \forall v)(\langle z, v\rangle \in C \wedge \langle z, u\rangle \in C \Rightarrow v = u) \Rightarrow \exists x \forall y(y \in x \Leftrightarrow \exists z(z \in a \wedge \langle z, y\rangle \in C))$, i.e. if C is a functional class and a is a set, then $\{C(z) : z \in a\}$, the image of a under C, is also a set. The intended interpretation of the second-order variables ranges over arbitrary subsets of the domain.

Show that if λ is a cardinal such that $V_\lambda \models ZFC^2$, then λ is strongly inaccessible.

2

(i) Define or state (as appropriate) any two of the following three italicized concepts or assertions:

(a) the family $\{A_{\alpha,n} : \alpha < \omega_1, n < \omega\}$ of sets is an *Ulam matrix* on ω_1;

(b) *König's Theorem* on cofinality and cardinal exponentiation;

(c) the binary predicate $x \in L_\alpha$.

(ii) Show that $\aleph_1 \leqslant \mathfrak{b} \leqslant \mathfrak{a} \leqslant 2^{\aleph_0}$, where $\mathfrak{a}$ and $\mathfrak{b}$ are the almost disjointness number and the bounding number associated with the quasi-order $\preceq^*$ on the set ${}^\omega\omega$, defined as follows:

$$f \preceq^* g \Leftrightarrow f(n) \leqslant g(n) \text{ for all but finitely many } n \in \omega.$$

(iii) (a) Prove that $ZF \vdash \varphi^L$ where φ is an instance of the axiom of Separation.

(b) Suppose that $\mathbb{A} = (A, \in)$ is a standard model of ZF. Stating precisely any absoluteness results to which you may appeal, show

$$L^{\mathbb{A}} = \bigcup_{\alpha \in A} L_\alpha.$$

(c) Deduce that L is the smallest standard model of ZF containing all the ordinals. (You may assume without proof that $L \models ZF$.)

(iv) Suppose $\kappa = cf(\kappa) > \aleph_0$. Prove that there exists a family $\{S_\alpha \subseteq \kappa : \alpha < \kappa\}$ of pairwise disjoint stationary sets such that $\kappa = \bigcup_{\alpha<\kappa} S_\alpha$.

3

(i) Define carefully any <u>two</u> of the following three italicized expressions:

(a) the *Gimel Hypothesis*;

(b) the assertion MA_κ, where κ is a cardinal;

(c) the tree $\mathbb{T} = (T, <_{\mathbb{T}})$ is *well-pruned*.

(ii) Show that if $\delta > 0$ is a limit ordinal, then $cf(\beth_\delta) = cf(\delta)$.

(iii) Show that the Gimel function $\gimel(\kappa)$ determines the class functions $\kappa \to 2^\kappa$ and $(\kappa, \lambda) \to \kappa^\lambda$.

(iv) (a) Suppose that $\mathbb{T} = (T, <_{\mathbb{T}})$ is a κ-tree. Show that $\mathbb{T}$ has a well-pruned κ-subtree (i.e. a subtree that is a κ-tree and is well-pruned).

(b) Deduce that if there is an $\aleph_1$-Suslin tree and $2^{\aleph_0} > \aleph_1$, then Martin's Axiom fails.

4

(i) Define carefully any <u>two</u> of the following three italicized expressions:

(a) the *partition relation* $\kappa \to (\lambda)^\alpha_\beta$ for cardinals $\alpha, \beta, \kappa, \lambda$;

(b) the *prediction principle* $\clubsuit_S$ for a stationary subset $S \subseteq \omega_1$;

(c) the tree $\mathbb{T} = (T, <_{\mathbb{T}})$ has *unique limits*.

(ii) Suppose that $\mathbb{T} = (T, <_{\mathbb{T}})$ is a normal $(\aleph_1, \aleph_1)$-tree with no uncountable anti-chains. Prove that $\mathbb{T}$ is $\aleph_1$-Suslin.

(iii) Show that $\diamondsuit$ implies that Suslin's Hypothesis fails. (It suffices to outline the principal elements in the construction, but you should provide full details of the verification of the countable chain condition.)

(iv) Suppose $\kappa = cf(\kappa) > \aleph_0$ and $\{A_\alpha : \alpha < \kappa\}$ is a κ-filtration of a set A of cardinality κ. Prove the following variant of Fodor's Lemma: if S is a stationary subset of κ and $f : S \to A$ is a function such that for all $\alpha \in S$, $f(\alpha) \in A_\alpha$, then there exists a stationary $S' \subseteq S$ such that $f \restriction S'$ is constant.

5

(i) Let G be generic in the forcing $\mathbb{P}$ over a countable transitive model $\mathbb{M}$. Define or state carefully any <u>two</u> of the following three italicized concepts or assertions:

(a) the set τ is a $\mathbb{P}$-*name*;

(b) the *Truth Lemma* for the generic extension $\mathbb{M}[G]$;

(c) the forcing $\mathbb{P}$ *preserves cofinalities.*

(ii) Suppose that the Axiom of Choice AC holds in the countable transitive model $\mathbb{M}$. Suppose G is generic in $\mathbb{P}$ over $\mathbb{M}$. Prove that $\mathbb{M}[G] \models AC$.

(iii) Show that $ZFC + 2^{\aleph_0} = 2^{\aleph_1} = \aleph_3$ is consistent relative to $ZFC + GCH$.

(iv) *Kurepa's Hypothesis* states that there exists an $\aleph_1$-tree with at least $\aleph_2$ paths (i.e. maximal chains of cardinality $\aleph_1$).

Suppose that κ is strongly inaccessible in the countable transitive model $\mathbb{M}$.

Let H be generic in the forcing $Lv(\kappa)$ over $\mathbb{M}$, where $Lv(\kappa)$ is the Levy collapse

$$\{p : p \text{ is a finite function, } dom(p) \subseteq \kappa \times \omega, \text{ and } (\forall \langle \alpha, n\rangle \in dom(p))(p(\alpha, n) \in \alpha)\}$$

with the partial ordering $p \leqslant q$ iff $p \subseteq q$.

(a) What is the cardinality of a maximal antichain in $Lv(\kappa)$?

(b) Prove that $\kappa = \aleph_1^{\mathbb{M}[H]}$.

(c) By considering the full binary tree of height κ in $\mathbb{M}$ or otherwise, determine whether Kurepa's Hypothesis is true in the generic extension $\mathbb{M}[H]$.

6

(i) Let $\mathbb{P}$ be a forcing in the countable transitive model $\mathbb{M}$. Define or state carefully any <u>two</u> of the following three italicized concepts or results:

(a) the forcing $\mathbb{P}$ *collapses* the cardinal κ;

(b) the set G is *generic* in $\mathbb{P}$ over $\mathbb{M}$;

(c) the Δ-*System Lemma*.

(ii) Suppose that G is generic in the forcing $\mathbb{P}$ over the countable transitive model $\mathbb{M}$, and $\mathbb{P}$ is κ-complete in $\mathbb{M}$, where $\kappa = cf(\kappa) > \aleph_0$. Let $\alpha \in Ord, \alpha < \kappa, B \in M$.

Show that $({}^{\alpha}B)^{\mathbb{M}} = ({}^{\alpha}B)^{\mathbb{M}[G]}$, i.e. if $f : \alpha \to B, f \in \mathbb{M}[G]$, then $f \in \mathbb{M}$.

(iii) Prove that $ZFC + GCH$ does not imply $V = L$.

(iv) (a) Show that there exist functions $\{e_\alpha : \alpha < \omega_1\}$ such that each $e_\alpha : \alpha \to \omega$ is injective and for $\beta < \alpha, e_\beta$ and $e_\alpha \restriction \beta$ are identical at all but finitely many points.

(b) Considering the tree $\mathbb{T} = \{e_\alpha \restriction \beta : \beta \leqslant \alpha < \omega_1\}$ with the functions from part (a) partially ordered by $f \prec g$ iff g is a proper extension of f, or adducing any other valid reasons, prove that there exists an Aronszajn tree, i.e. an $\aleph_1$-tree with no uncountable branches.

END OF PAPER