UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 2 June, 2014 1:30 pm to 4:30 pm

PAPER 18

CATEGORY THEORY

*Attempt no more than **FIVE** questions.*

*There are **EIGHT** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Define the terms *monomorphism*, *epimorphism*, *regular epimorphism* and *isomorphism*. In a category with products, show that the diagonal $(1,1)\colon A \to A \times A$ defined by $\pi_1(1,1) = 1_A$ and $\pi_2(1,1) = 1_A$ is a monomorphism.

(b) We define a *strong epimorphism* to be any morphism $g\colon C \to D$ in a category $\mathcal{C}$ such that for any commutative diagram

$$\begin{tikzcd} C \arrow[r, "k"] \arrow[d, "g"'] & A \arrow[d, "f"] \\ D \arrow[r, "l"'] & B \end{tikzcd}$$

with f monic, there is a (necessarily unique) lifting $t\colon D \to A$ satisfying $tg = k$ and $ft = l$. (Note that here we do *not* assume that g is an epimorphism.)

Show that every regular epimorphism is a strong epimorphism. Show that any morphism which is both a monomorphism and a strong epimorphism is an isomorphism. Show that if the composite gf is a strong epimorphism, then g is also a strong epimorphism. Deduce that if $f = ig$ is a strong epimorphism and i is monic, then i is an isomorphism.

(c) Let $\mathcal{C}$ be a category with binary products. Show that any strong epimorphism is indeed an epimorphism.

2

(a) State the Yoneda Lemma and explicitly give the isomorphism in both directions.

(b) Let $F, G\colon \mathcal{C} \to \mathrm{Set}$ be functors. Prove that a natural transformation $\alpha\colon F \to G$ is epic in the functor category $[\mathcal{C}, \mathrm{Set}]$ if and only if each component α_A is epic in Set.

[You may assume that $\mathrm{ev}_A\colon [\mathcal{C}, \mathcal{D}] \to \mathcal{D}$ preserves all colimits which exist in $\mathcal{D}$, but should prove all other statements carefully.]

(c) Define a projective object in a category $\mathcal{C}$. Prove that a coproduct of projective objects is projective.

(d) Let $\mathcal{C}$ be locally small. Show that a representable functor $\mathcal{C}(A, -)$ is projective in $[\mathcal{C}, \mathrm{Set}]$.

3

Let $\mathcal{C}$ be a locally small category and $U\colon \mathcal{C} \to \mathrm{Set}$ a functor.

(a) Define what it means for the functor U to be representable. Prove that the identity functor $1_{\mathrm{Set}}\colon \mathrm{Set} \to \mathrm{Set}$ is representable.

(b) Prove that, in the category Set, any set is a coproduct of copies of the one-element set 1.

(c) Prove: U has a left adjoint $\Rightarrow$ U is representable.

(d) Prove: U is representable $\Rightarrow$ U preserves limits.

(e) If $\mathcal{C}$ has small coproducts, prove: U is representable $\Rightarrow$ U has a left adjoint.

4

Consider two functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ and natural transformations $\eta\colon 1_{\mathcal{C}} \to GF$, $\varepsilon\colon FG \to 1_{\mathcal{D}}$.

(a) Prove that F is left adjoint to G (in symbols $F \dashv G$) if and only if η and ε satisfy the triangular identities

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\varepsilon_F"] \\ & F \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\varepsilon"] \\ & G \end{tikzcd}$$

(b) Suppose now that $G \xrightarrow{\eta_G} GFG \xrightarrow{G\varepsilon} G$ is the identity on G, and show that the composite $F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon_F} F$ is an idempotent (in the functor category $[\mathcal{C}, \mathcal{D}]$). Show that G has a left adjoint if and only if this idempotent splits.

[Recall that an *idempotent* is a morphism $e\colon E \to E$ with $ee = e$, and such an idempotent *splits* if there exist $f\colon E \to F$ and $g\colon F \to E$ with $e = gf$ and $fg = 1_F$.

Hint: For $\Rightarrow$ of the last statement, consider $H \dashv G$ with unit $\psi\colon 1 \to GF$ and counit $\varphi\colon HG \to 1$, and use ψ, φ, η and ε to build candidate natural transformations $F \to H \to F$.]

5

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and consider a functor $G\colon \mathcal{D} \to \mathcal{C}$.

Define, for each object A of $\mathcal{C}$, the category $(A \downarrow G)$.

State and prove the General Adjoint Functor Theorem.

[You may assume results about limits in $(A \downarrow G)$ and standard results about adjoints (other than any Adjoint Functor Theorem), provided they are clearly stated.]

6

(a) Define a *monad* $\mathbb{T}$ on a category $\mathcal{C}$, and the *category of* $\mathbb{T}$*-algebras*. Describe the list monad on Set and determine its category of algebras.

(b) Show that the forgetful functor from the category $\mathcal{C}^{\mathbb{T}}$ of $\mathbb{T}$-algebras to $\mathcal{C}$ has a left adjoint and that the adjunction induces the monad $\mathbb{T}$.

(c) Let $\mathbb{T}$ be a monad on $\mathcal{C}$. Prove carefully that any $\mathbb{T}$-algebra is a coequaliser of a diagram of free algebras.

[You may assume any standard results from the course provided they are clearly stated.]

7

(a) Define what an *isomorphism* in a category $\mathcal{C}$ is, and when a category $\mathcal{C}$ is a *groupoid*. Show that a morphism $f\colon A \to B$ with $gf = 1_A$ and $fh = 1_B$ for $g, h\colon B \to A$ must be an isomorphism.

(b) Define a *preadditive* category $\mathcal{A}$.

Let
$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "g"'] & B \arrow[l, "r" description] \end{tikzcd}$$
be a reflexive pair in the preadditive category $\mathcal{A}$, i.e. $fr = gr = 1_B$. Prove that this has the structure of an internal groupoid: that is, for any object C of $\mathcal{A}$, $\mathcal{A}(C, B)$ is the set of objects of a groupoid with set of morphism $\mathcal{A}(C, A)$ and domain and codomain given by composition with f and g respectively.

[*Hint: For a composable pair of morphism $a, b \in \mathcal{A}(C, A)$, their composition is given by $a + b - rga \in \mathcal{A}(C, A)$. You should however check that this has the correct domain and codomain.*]

8

(a) Let $\mathcal{C}$ be a pointed category. Define the *cokernel* of a morphism $f\colon A \to B$ in $\mathcal{C}$. Now let $\mathcal{C}$ be pointed with cokernels. Given a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[d, "c"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \end{tikzcd}$$

with $g = \mathrm{coker} f$ and $g' = \mathrm{coker} f'$, show that if the left hand square is a pushout, then c is an isomorphism.

(b) Prove that in an abelian category, a morphism $f\colon A \to B$ is an epimorphism if and only if its cokernel is zero. Deduce that in an abelian category, pushouts reflect epimorphisms.

(c) In an abelian category $\mathcal{A}$, consider a square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

with g an epimorphism. Prove that if this square is a pullback, then it is also a pushout. Deduce that in an abelian category, pullbacks preserve epimorphisms. Deduce also that, in an abelian category, any epimorphism is the coequaliser of its kernel pair.

END OF PAPER