UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 4 June, 2014 1:30 pm to 4:30 pm

PAPER 17

COMPLEX MANIFOLDS

Attempt no more than ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Define the *Dolbeault cohomology* groups $H^{p,q}_{\bar\partial}(X)$ of a complex manifold X. Also define what it means for the sequence

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

of sheaves of abelian groups on X to be *exact*. Write down the associated long exact sequence of Čech cohomology groups.

(b) State and prove Dolbeault's Theorem.

(c) Using this, compute $H^{p,q}_{\bar\partial}(\mathbb{P}^1 \times \mathbb{C})$ for all $p \in \{0, 2\}$ and all q.

[General properties of Cech cohomology may be used without proof if stated clearly and you may assume that $H^{p,q}_{\bar\partial}(\mathbb{C}^r \times (\mathbb{C}^*)^s) = 0$ for all $p, r, s \geqslant 0$ and all $q \geqslant 1$.]

2

(a) Define what it means for E to be a *holomorphic vector bundle* on a complex manifold X. Define the Picard group $\mathrm{Pic}(X)$ and write down the *Picard group* of $\mathbb{P}^1$.

(b) Let $\pi : E \to X$ be a holomorphic vector bundle of rank 2 on X. Show that

$$\mathbb{P}(E) = \{l : l \text{ is a one dimensional complex subspace of } \pi^{-1}(x) \text{ for some } x \in X\}$$

can be made into a complex manifold such that there is a holomorphic map $p : \mathbb{P}(E) \to X$ with the property that

$$p^{-1}(x) \cong \mathbb{P}^1 \tag{1}$$

for all $x \in X$. Define, without proof, a holomorphic line bundle L on $\mathbb{P}(E)$ with the property that

$$L|_{p^{-1}(x)} \cong \mathcal{O}_{\mathbb{P}^1}(1)$$

under the isomorphism in (1).

(c) Finally, prove the map

$$\mathrm{Pic}(X) \times \mathbb{Z} \to \mathrm{Pic}(\mathbb{P}(E))$$

given by

$$(M, n) \to p^*M \otimes L^{\otimes n},$$

where L is the line bundle above, is well-defined and injective.

3

Suppose X is a compact complex manifold and ω is a Kähler metric on X. Define the Lefschetz operator L, the associated operator Λ, the operator $\bar{\partial}^*$ and the Laplacian $\Delta_{\bar{\partial}}$. Prove that a form α satisfies $\Delta_{\bar{\partial}}\alpha = 0$ if and only if $\bar{\partial}\alpha = 0$ and $\bar{\partial}^*\alpha = 0$.

Assuming the identity $[\bar{\partial}^*, L] = i\partial$ show

$$[L, \Delta_{\bar{\partial}}] = 0$$

and prove that L induces a map

$$\phi_{\omega,k} : H^{p,q}_{\bar{\partial}}(X) \to H^{p+k,q+k}_{\bar{\partial}}(X)$$

for all $k \geqslant 1$. Now suppose that f is a smooth function on X such that

$$\omega' = \omega + i\partial\bar{\partial} f$$

is also a Kähler-form. Show that $\phi_{\omega,k} = \phi_{\omega',k}$ for all k.

4

Let X be a complex manifold and g be a Riemannian metric on the underlying smooth manifold. Define what it means for g to be *compatible* with the complex structure, and define the *associated fundamental form* ω to g. Define what it means for g to be *Kähler*, and explain why any Riemannian metric on a complex manifold of dimension 1 that is compatible with the complex structure is Kähler.

Now let g be a Riemannian metric on a complex manifold X of dimension n compatible with the complex structure and ω be the associated fundamental form. Prove that the following are equivalent:

(a) $d\omega = 0$.

(b) For any point $x \in X$ there exist holomorphic coordinates $z_1, \ldots, z_n$ centred at x such that locally

$$\omega = \frac{i}{2} \sum_{i,j=1}^{n} h_{ij} dz_i \wedge d\overline{z}_j$$

where $h_{ij} = \delta_{ij} + O(|z|^2)$.

(c) For any point $x \in X$ there exists an open $x \in U \subset X$ and a smooth real function f defined on U such that

$$\omega|_U = i\partial\bar{\partial} f.$$

Now let $X = \mathbb{C}$ and suppose that

$$\omega = i\partial\bar{\partial} f$$

for some smooth function f with the property that

$$f(e^{i\theta} z) = f(z) \quad \text{for all } \theta \in [0, 2\pi].$$

By considering the Taylor series expansion of f in z and $\overline{z}$, or otherwise, show that the formula

$$u(t) = f(z) \quad \text{where } t = \ln |z|^2$$

gives a well-defined smooth function $u \colon \mathbb{R} \to \mathbb{R}$, and that ω is the associated form of a Kähler metric if and only if $u''(t) > 0$ for all $t \in \mathbb{R}$ and $\lim_{t \to -\infty} u''(t) e^{-t} > 0$.

5

Let $\pi \colon E \to X$ be a holomorphic vector bundle on a complex manifold X. Define what it means for h to be a *hermitian metric* on E, what it means for D to be a connection on E and what it means for D to be *compatible* with h. Prove that if D is compatible with h then there exists a local frame $s_1, \ldots, s_r$ for E such that the connection matrix of D is skew-hermitian.

Show that there exists a naturally defined holomorphic vector bundle $E^* \to X$ whose fibre over any $x \in X$ is the dual space $(E_x)^*$. Using this, explain how h can be thought of as a smooth section H of $(E \otimes \overline{E})^*$ where $\overline{E}$ is a complex vector bundle that should also be defined.

Finally, show that D induces a connection D_0 on $(E \otimes \overline{E})^*$, and prove that D is compatible with h if and only if

$$D_0(H) = 0.$$

END OF PAPER