UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 9 June, 2014 1:30 pm to 4:30 pm

PAPER 16

SYMPLECTIC TOPOLOGY

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Explain how the Euler–Lagrange equations in Lagrangian mechanics are derived from a variational principle. (Include a proof.)

(b) Describe how Hamiltonian mechanics emerges from Lagrangian mechanics. (Include a proof.)

2

(a) What is a symplectic manifold?

(b) For which n does the n-sphere S^n admit a symplectic structure? Justify your answer.

(c) How many symplectic structures are there on the Möbius strip? Justify your answer.

(d) What is a symplectic vector field on a symplectic manifold? What is a Hamiltonian vector field on a symplectic manifold? Consider the symplectic 2-torus $(\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2, \omega = dx \wedge dy)$. Is the vector field $\frac{\partial}{\partial x}$ on $\mathbb{T}^2$ symplectic? Is it Hamiltonian? Justify your answer.

(e) Let (M, ω) be a $2n$-dimensional compact symplectic manifold without boundary, and let $H \in C^\infty(M)$ be a smooth function. What is the Hamiltonian flow ϕ_H^t of H? Prove that the Hamiltonian flow ϕ_H^t preserves the symplectic volume $\frac{1}{n!}\,\omega^n$.

3

(a) Briefly explain Moser's trick.

(b) State and prove Darboux's theorem.

4

(a) Briefly explain how the cotangent bundle T^*L of a manifold L can be equipped with a canonical symplectic form ω_{can}.

(b) What is a Lagrangian submanifold of a symplectic manifold? Show that the graph of a 1-form σ on a manifold L, where σ is viewed as a section $\sigma : L \to T^*L$, is Lagrangian in $(T^*L, \omega_{\mathrm{can}})$ if and only if σ is closed.

(c) Let (M, ω) be a compact symplectic manifold without boundary, and assume that $H^1_{\mathrm{dR}}(M) = 0$. Prove that any symplectomorphism of (M, ω) that is sufficiently C^1-close to the identity has at least two fixed points. [You may use any result from the lecture without proof, provided you state it clearly.]

5

(a) What does it mean for a Lie group action on a symplectic manifold to be Hamiltonian?

(b) Define the Fubini–Study form on projective space $\mathbb{C}\mathrm{P}^n$.

(c) Show how the Fubini–Study form on $\mathbb{C}\mathrm{P}^n$ arises from a Hamiltonian group action on $\mathbb{C}^{n+1}$. [You may use without proof the Marsden–Weinstein theorem, provided you state it clearly.]

6

(a) What is an ω-compatible almost complex structure on a symplectic manifold (M, ω)?

(b) What is a J-holomorphic curve?

(c) Define the energy of a J-holomorphic curve. State and prove the energy identity for J-holomorphic curves.

(d) Let (Σ, j) be a compact connected Riemann surface without boundary. Let J be an ω_0-compatible almost complex structure on $(\mathbb{R}^{2n}, \omega_0)$, where ω_0 denotes the standard symplectic form. Does there exist a nonconstant J-holomorphic curve $u : \Sigma \to \mathbb{R}^{2n}$? Justify your answer.

END OF PAPER