UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 29 May, 2014 1:30 pm to 4:30 pm

PAPER 15

DIFFERENTIAL GEOMETRY

Attempt no more than ***THREE*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State the defining properties of the exterior derivative d and show, using local coordinates, that these properties uniquely determine d.

Define the de Rham cohomology of a manifold.

Prove that the 1st de Rham cohomology of the 2-dimensional sphere is $H^1_{dR}(S^2) = \{0\}$. By considering the quotient map $S^2 \to \mathbb{R}P^2 \cong S^2/\pm 1$, or otherwise, determine the de Rham cohomology groups of $\mathbb{R}P^2$.

[*The Poincaré lemma may be assumed provided it is accurately stated. You may assume that the antipodal map of S^2 induces, via pull-back, the multiplication by -1 on $H^2_{dR}(S^2)$.*]

2

Define what is meant by a Lie group.

Recall that the complex symplectic group $Sp(n)$ may be defined as the subgroup of unitary matrices $A \in U(2n)$ satisfying $AJA^t = J$, where A^t denotes the transpose of A and $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ with I the $n \times n$ identity matrix. Explain why $e^{-JBJ} = -Je^B J$ holds for each $n \times n$ complex matrix B. Show that $Sp(n)$ is a manifold, by constructing an appropriate family of charts. Show further that $Sp(n)$ is a Lie group and find its dimension.

Show that $Sp(1)$ is diffeomorphic to S^3.

[*Standard results on the exponent and logarithm of matrices may be used without proof if these are accurately stated.*]

3

Show that every real vector bundle over a manifold M can be given an inner product on the fibres, varying smoothly with the fibres (with respect to any local trivialization).

Define what is meant by a bundle morphism $E' \to E''$ covering the identity map of M, where E_i, $i = 1, 2$ are vector bundles over M. Let $\Gamma(E_i)$ denote the space of smooth sections of E_i. Show that every map $\alpha : \Gamma(E') \to \Gamma(E'')$ which is linear over $C^\infty(M)$ is induced by some bundle morphism $F : E' \to E''$, i.e. $\alpha(s) = F \circ s$ for each $s \in \Gamma(E')$.

Are the vector bundles TM and T^*M isomorphic for every manifold M? Justify your answer.

[*Existence of partition of unity on M may be assumed provided the result is accurately stated.*]

4

Define the Levi–Civita connection on a Riemannian manifold. Prove that every Riemannian manifold (M, g) admits a unique Levi–Civita connection.

Let (M, g_M) and (N, g_N) be two Riemannian manifolds. Explain what is meant by the product Riemannian metric $g_M \oplus g_N$ on $M \times N$. Let X be a vector field on M and Y a vector field on N. Prove carefully that, considering X and Y as vector fields on $M \times N$ independent of, respectively, the N coordinates and the M coordinates, we have $D_X Y = 0$ where D is the Levi-Civita connection of $g_M \oplus g_N$.

[*You may assume that* $[X, Y] = 0$.]

5

Let (M, g) be an oriented Riemannian manifold. Define the volume form ω_g of g showing that ω_g is well-defined.

Define the Hodge star operator $*$ and compute its square for differential p-forms on M. Show that $\int_M (-f * d * \alpha)\omega_g = \int_M g(df, \alpha)\, \omega_g$, for every compactly supported smooth function f and 1-form α on M.

Define the Laplace–Beltrami operator Δ and the harmonic forms on M. Prove that a differential form β on M is harmonic if and only if $*\beta$ is so. Prove the identity $\Delta(f_1 f_2) = f_2 \Delta f_1 + f_1 \Delta f_2 - 2g(df_1, df_2)$, for functions $f_1, f_2 \in C^\infty(M)$.

State the Hodge decomposition theorem. Prove that if M is a compact connected oriented n-dimensional manifold, then its n-th de Rham cohomology $H^n_{dR}(M)$ is 1-dimensional.

[*Stokes' theorem may be used without proof.*]

END OF PAPER