UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS **Part III**

Tuesday, 3 June, 2014 9:00 am to 12:00 pm

# PAPER 14

# 3-MANIFOLDS

*Attempt **Question 5** and no more than **TWO** of Questions 1-4.*

*There are **FIVE** questions in total.*

*Each of Questions 1-4 is worth 25 marks;*
*each part of Question 5 is worth 10 marks.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

What is meant by a Heegaard splitting of a 3-manifold $Y$? Given a triangulation of a closed, connected, orientable 3-manifold $Y$, explain how to find a Heegaard splitting.

Let $K$ be a null-homologous knot in an oriented 3-manifold $Y$. What is meant by $p/q$ Dehn surgery on $Y$? If $Y$ is obtained by integral Dehn surgery on a link $L \subset S^3$, show that $Y$ is the boundary of a compact orientable 4-manifold. Deduce that if $Y$ is obtained by an arbitrary Dehn surgery on a link $L \subset S^3$, then it bounds a compact orientable 4-manifold.

**2**

Let $\gamma$ be a simple closed curve on a closed oriented surface $S$. What is meant by the right-handed Dehn twist $\tau_\gamma$? Express the action of $(\tau_\gamma)_* : H_1(S) \to H_1(S)$ in terms of $[\gamma]$.

Determine whether the following statements are true or false. Justify your answers.

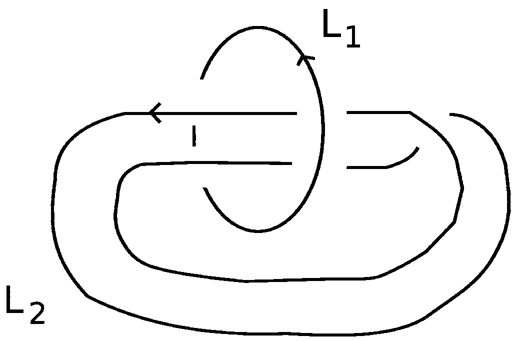
1. Any element of $MCG^+(T^2)$ is a composition of positive Dehn twists. [*Hint: Consider* $(\tau_\ell \tau_m)^6$.]
2. Let $L \subset S^1 \times S$ be a link each of whose components have the form $L_i = p_i \times \gamma_i$, where $\gamma_i$ is a simple closed curve on $S$ and $p_i$ is a point in $S^1$. Let $\ell_i = p_i' \times \gamma_i$ be a longitude for $L_i$. Then the manifold obtained by doing $+1$ surgery (with respect to this choice of longitude) on each component of $L_i$ fibres over $S^1$.

**3**

Suppose $L \subset S^3$ is the oriented link shown the figure below. Write $X = S^3 - \nu(L)$, and let $m_1$ and $\ell_1$ be the standard meridian and longitude for $L_1$. If $K \subset S^3$ has meridian $m$ and longitude $\ell$, let $Y = (S^3 - \nu(K)) \cup_{T^2} X$, where we identify $m_1$ with $\ell$ and $\ell_1$ with $m$.

Show that $Y$ is the complement of a knot $C(K) \subset S^3$.

Express the Alexander polynomial $\Delta(C(K))$ in terms of $\Delta(K)$.




**4**

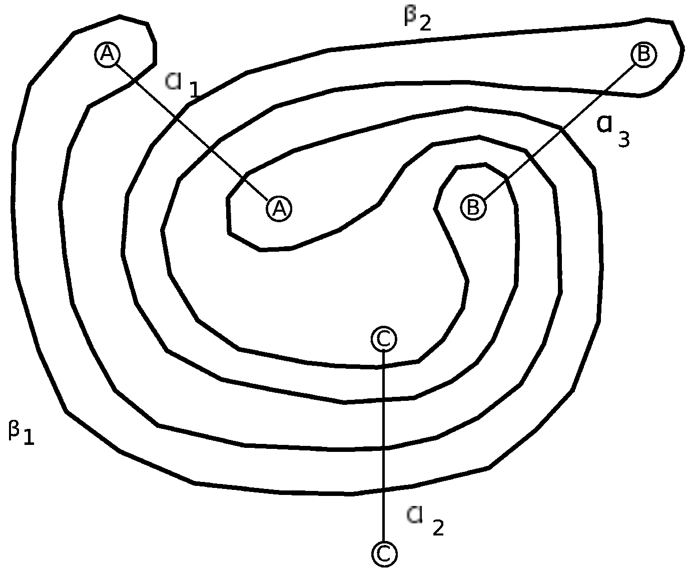
Let $Y$ be a compact orientable 3-manifold whose boundary is a union of tori. Define the Thurston norm $\|\cdot\|$ on $H_2(Y, \partial Y)$.

If $Y$ is closed and connected, can any class $x \in H_2(Y)$ be represented by an embedded connected surface $S \subset Y$? Justify your answer.

Now suppose $X$ is a closed 3-manifold with $H_*(X) \cong H_*(S^3)$, that $K \subset X$ is a knot, and that $Y = X - \nu(K)$. Let $Z$ be the manifold obtained by doing 0-framed Dehn surgery on $K$. Suppose $y$ generates $H_2(Y, \partial Y)$, and that $z$ generates $H_2(Z)$. Show that if $\|y\| > 0$, then $\|z\| \leqslant \|y\| - 1$. Give an example where the inequality is strict. (Hint: take $Z = S^1 \times S^2$.)

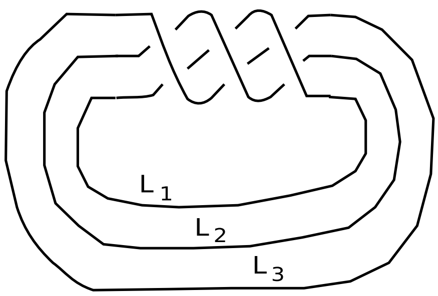
**5**

1. Let $Y$ be the 3-manifold represented by the generalized Heegaard diagram below. Explain why $Y = S^3 - \nu(L)$ for some link $L$. Find a presentation for $\pi_1(Y)$.




2. Let $\widetilde{Y}$ be the universal abelian cover of $Y$. Describe $C_*^{cell}(\widetilde{Y})$ as a chain complex over $R = \mathbb{Z}[H_1(Y)]$. Compute $\Delta(L)$.

   For the rest of the problem, you may take as given that $L$ is the link shown below.

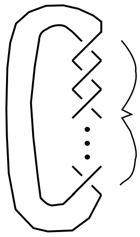



3. What is the dual Thurston polytope $B_T(Y)$? Find embedded surfaces $S_1, \ldots, S_n \subset Y$ such that $B_T(Y)$ is cut out by the equations

$$|\alpha \cdot S_i| \leqslant c(S_i)(i = 1 \ldots, n).$$

4. Using your answer to part 2, compute $\Delta(L(n))$, where $L(n)$ is the two component link shown below.

L(n) =



2n crossings

5. Let $Z$ be the manifold obtained by doing $1/2$ surgery on $L_1$, $1/3$ surgery on $L_2$, and $1/5$ surgery on $L_3$. What is $H_1(Z)$? Identify the manifold obtained by doing $-1$ surgery on $L_1$, 0 surgery on $L_2$, and $+5$ surgery on $L_3$.

5. Let $Y$ be the manifold obtained by doing $1/2$ surgery on $L_1$, $1/3$ surgery on $L_2$, and $1/5$ surgery on $L_3$. What is $H_1(Y)$? Identify the manifold obtained by doing $-1$ surgery on $L_1$, 0 surgery on $L_2$, and $+5$ surgery on $L_3$.

## END OF PAPER