UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 3 June, 2014 1:30 pm to 4:30 pm

PAPER 13

ALGEBRAIC GEOMETRY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

For $\mathcal{F}$ a presheaf of abelian groups on a topological space X, describe the construction of the sheafification $\mathcal{F}^+$ and the corresponding morphism of presheaves $\theta : \mathcal{F} \to \mathcal{F}^+$. State the universal property satisfied by this morphism, and in the case when $\mathcal{F}$ is itself a sheaf show that θ is an isomorphism.

Suppose now that $f : X \to Y$ is a continuous map of topological spaces, that $\mathcal{F}$ is a sheaf of abelian groups on X and $\mathcal{G}$ a sheaf of abelian groups on Y. Describe briefly the construction of the sheaves $f_*\mathcal{F}$ on Y and $f^{-1}\mathcal{G}$ on X.

An *f-morphism* $\phi : \mathcal{G} \to \mathcal{F}$ is defined by giving homomorphisms of abelian groups $\phi(U) : \mathcal{G}(U) \to \mathcal{F}(f^{-1}U)$ for all U open in Y which are compatible with restrictions, namely for $V \subset U$ an inclusion of open sets and $\sigma \in \mathcal{G}(U)$, we have $\phi(V)(\sigma|_V) = (\phi(U)(\sigma))|_{f^{-1}V}$. Show that such an f-morphism ϕ induces homomorphisms on stalks $\phi_{f(P)} : \mathcal{G}_{f(P)} \to \mathcal{F}_P$ for all $P \in X$.

Show that there is a natural f-morphism $\theta : \mathcal{G} \to f^{-1}\mathcal{G}$ with the property that any f-morphism $\phi : \mathcal{G} \to \mathcal{F}$ determines a unique morphism $\psi : f^{-1}\mathcal{G} \to \mathcal{F}$ of sheaves on X with $\phi(U) = \psi(f^{-1}U) \circ \theta(U)$ for all U open in Y.
[*Hint: To define the image of a section $s \in (f^{-1}\mathcal{G})(V)$, you may need to glue together certain sections of $\mathcal{F}$ over some open cover of V.*]

Deduce that there is a natural bijection between $\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F})$ (morphisms of sheaves on X) and $\mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$.

2

For an affine variety V over an algebraically closed field k, describe briefly (without proofs) one of the two possible constructions for the sheaf of regular functions $\mathcal{O}_V$. Define briefly what is meant by an algebraic variety X over an algebraically closed field k (you need not define the Zariski topology on $X \times X$) and by a morphism of such varieties. Show that affine varieties are algebraic varieties, and that regular maps between affine varieties give rise to morphisms of varieties in the sense just defined.

Let $\phi : X \to Y$ be a morphism of varieties, and suppose that Y can be covered by open sets U_i such that the induced morphism $\phi^{-1}(U_i) \to U_i$ is an isomorphism of varieties for each i; show that ϕ is an isomorphism.

From now on, we suppose that X is a variety over an algebraically closed field k, and A denotes the k-algebra $\Gamma(X, \mathcal{O}_X)$.

For Y an affine variety over k with $B = \Gamma(Y, \mathcal{O}_Y)$, show that any k-algebra homomorphism from B to A induces a morphism of varieties from X to Y.

Now suppose that $f \in A$ and $X_f := \{x \in X \ : \ f(x) \neq 0\}$; show that $\Gamma(X_f, \mathcal{O}_X) = A_f$. Suppose that there is a finite set $f_1, \ldots, f_r$ of elements of A which generate the unit ideal, and that X_{f_i} is affine for all i; prove that X is itself affine.

3

For $(X, \mathcal{O}_X)$ a variety, define what it means for an $\mathcal{O}_X$-module $\mathcal{F}$ to be *quasi-coherent*. Let X be an affine variety with coordinate ring A and basic open sets $D(f) = \{x \in X : f(x) \neq 0\}$ for $f \in A$. For each A-module M, recall that an $\mathcal{O}_X$-module $\tilde{M}$ on X may be defined with the property that $\tilde{M}(D(f)) \cong M_f$ for all $f \in A$, with stalks given by the localizations of M at the maximal ideals of A. Prove that an $\mathcal{O}_X$-module on the affine variety X is quasi-coherent if and only if it is of the form $\tilde{M}$ for some A-module M. Deduce that taking global sections of quasi-coherent sheaves on an affine variety is an exact functor.

For U an open subset of a topological space X and $\mathcal{F}$ a sheaf of abelian groups on X, define the sheaf ${}_U\mathcal{F}$ on X and the morphism $\mathcal{F} \to {}_U\mathcal{F}$. With cohomology groups constructed via flabby resolutions, suppose $\mathcal{B}$ is a basis of open sets in X, closed under finite intersections, with $H^j(V, \mathcal{F}|_V) = 0$ for $0 < j < i$ and for all $V \in \mathcal{B}$; state (without proof) the *locally vanishing principle* for elements of $H^i(X, \mathcal{F})$. If $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module on an affine variety X, prove that $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

[Properties of the sheafification functor and standard results from Commutative Algebra should be assumed in this question.]

4

Given an open cover $\mathcal{U}$ of a topological space X, and a sheaf $\mathcal{F}$ of abelian groups on X, describe the construction of the Čech cohomology groups $\check{H}^i(\mathcal{U}, \mathcal{F})$. If X is an algebraic variety and $\mathcal{F}$ is a quasi-coherent sheaf on X, state (without proof) a criterion on $\mathcal{U}$ which ensures that $\check{H}^i(\mathcal{U}, \mathcal{F})$ is isomorphic to the cohomology group $H^i(X, \mathcal{F})$ (constructed for instance via flabby resolutions) for all $i \geqslant 0$. For $\mathcal{F}$ any quasi-coherent sheaf on $\mathbf{P}^n$, deduce that $H^i(\mathbf{P}^n, \mathcal{F}) = 0$ for all $i > n$.

Describe the construction of the invertible sheaves $\mathcal{O}_{\mathbf{P}^n}(m)$ on $\mathbf{P}^n$ (where $m \in \mathbf{Z}$). Prove that $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m))$ is isomorphic to the space of homogeneous polynomials of degree m in $k[X_0, \ldots, X_n]$ when $m \geqslant 0$, and is zero for $m < 0$.

For integers $d_0, d_1, \ldots, d_n \geqslant 0$, show that the homogeneous ideal $\langle X_0^{d_0}, X_1^{d_1}, \ldots, X_n^{d_n} \rangle$ of $k[X_0, \ldots, X_n]$ contains all homogeneous polynomials of degree $d = \sum_{i=0}^n d_i$. Using Čech cohomology, deduce that $H^n(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}) = 0$ for all $n > 0$.

Assuming standard general properties of cohomology and arguing by induction on n, prove (when $n > 0$) that

$$H^n(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(-r)) = 0$$

for all $r \leqslant n$.

Assuming that $\omega_0 = dx_1 \wedge \ldots \wedge dx_n$ is a generator for the regular n-forms on $\mathbf{A}^n$, show that the sheaf $\Omega^n_{\mathbf{P}^n}$ of regular n-forms on $\mathbf{P}^n$ is isomorphic to $\mathcal{O}_{\mathbf{P}^n}(-n-1)$. Say briefly why the above calculations are consistent with the statement of Serre duality on $\mathbf{P}^n$.

END OF PAPER