UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 30 May, 2014 1:30 pm to 4:30 pm

PAPER 12

ALGEBRAIC TOPOLOGY

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Compute the cohomology groups of the infinite-dimensional real projective space $\mathbb{RP}^\infty$. [*The ring structure in cohomology is not required.*]

For any topological space X, construct a homomorphism

$$\beta : H^n(X; \mathbb{Z}_m) \to H^{n+1}(X; \mathbb{Z}_m)$$

which fits into a long exact sequence

$$\cdots \to H^n(X; \mathbb{Z}_m) \to H^n(X; \mathbb{Z}_{m^2}) \to H^n(X; \mathbb{Z}_m) \xrightarrow{\beta} H^{n+1}(X; \mathbb{Z}_m) \to \cdots$$

Show that $\beta : H^n(\mathbb{RP}^\infty; \mathbb{Z}_2) \to H^{n+1}(\mathbb{RP}^\infty; \mathbb{Z}_2)$ is an isomorphism for n odd and is zero for n even.

2

Let X be a topological space. Define the cohomology with compact supports $H^*_{ct}(X)$. Compute $H^*_{ct}(\mathbb{R})$, where $\mathbb{R}$ has the Euclidean topology.

The *one-point compactification* of X is the set $X^+ = X \cup \{\infty\}$ comprising X and a disjoint point $\{\infty\}$, with a basis of open sets given by (i) the open sets in X and (ii) the unions $(X \backslash K) \cup \{\infty\}$, with $K \subset X$ compact. Prove that if X^+ is Hausdorff and locally contractible at ∞, i.e. admits a basis of contractible open neighbourhoods of ∞, then

$$H^*_{ct}(X) \cong \tilde{H}^*(X^+) \qquad (1)$$

Now let $X = \mathbb{Z} \times \mathbb{R} \subset \mathbb{R}^2$ with the Euclidean topology. Does (1) hold for X? Justify your answer.

3

Define the tautological complex line bundle over $\mathbb{CP}^n$, and prove that it is locally trivial. Hence, compute $H^*(\mathbb{CP}^n; \mathbb{Z})$ as a ring.

Let $n > 1$. Show that there is no continuous function $f : (S^2)^n \to S^2$ satisfying

1. f is invariant under permutations of the factors;
2. $f(x, \ldots, x) = x$ for every $x \in S^2$.

4

State the Poincaré duality theorem.

Let X be a closed connected orientable six-dimensional manifold. Prove that the Euler characteristic $\chi(X)$ is even. For each $k \in 2\mathbb{Z}$, construct a closed orientable six-manifold X_k with $\chi(X_k) = k$.

If X is a closed six-manifold which is not orientable, need $\chi(X)$ be even? Justify your answer.

5

Let $E \to X$ be a complex vector bundle of (complex) rank k over a compact space X. Let $\pi : \mathbb{P}(E) \to X$ denote the associated projective bundle, with fibre $\mathbb{CP}^{k-1}$. Show that the map

$$H^*(X) \oplus \cdots \oplus H^*(X) \longrightarrow H^*(\mathbb{P}(E)),\ (u_0, \ldots, u_{k-1}) \mapsto \sum \pi^*(u_i)t^i$$

is an isomorphism, so $H^*(\mathbb{P}(E))$ is the free $H^*(X)$-module with basis $\{1, t, t^2, \ldots, t^{k-1}\}$, where t is the Euler class of a line bundle $\mathcal{L}$ on $\mathbb{P}(E)$ which you should define. [*Hint: Imitate the proof of the Thom isomorphism theorem.*]

Deduce that $H^*(\mathbb{P}(E)) \cong H^*(X)[t]/I$, where $I = (f(t))$ is the ideal generated by a monic polynomial

$$f(t) = t^k - c_1(E)t^{k-1} + c_2(E)t^{k-2} + \cdots + (-1)^k c_k(E)$$

for uniquely defined classes $c_i(E) \in H^{2i}(X)$.

Finally, suppose $E \cong \oplus_{i=1}^k L_i$ is a direct sum of complex line bundles. By considering suitable sections $s_i : X \to \mathbb{P}(E)$, for $1 \leqslant i \leqslant k$, prove that

$$f(t) = \prod_{i=1}^{k}(t - e_i) \ \in H^*(X)[t]$$

where e_i is the Euler class of L_i.

END OF PAPER