UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 6 June, 2014 9:00 am to 11:00 am

PAPER 11

EXTREMAL AND PROBABILISTIC COMBINATORICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Write $b_e(A)$ for the cardinality of the edge-boundary of a subset A of the vertex set of the cube $Q^n = \{0,1\}^n$ (with 2^n vertices and $n2^{n-1}$ edges), and let $1 \leqslant k \leqslant n-1$.

(i) What is

$$f(2^k) = \min\{b_e(A) : \quad A \subset Q^n, \quad |A| = 2^k\}?$$

(ii) What is

$$g(2^k) = \min\{b_e(D) : \quad D \text{ is a down-set in } Q^n \text{ and } |D| = 2^k\}?$$

(iii) Determine

$$h(m) = \max\{b_e(D) : \quad D \text{ is a down-set in } Q^n \text{ with } |D| = m\}.$$

2

(i) State and prove the Erdős–Ko–Rado theorem.

(ii) Let $c_1, \ldots, c_n \geqslant 0$ be such that $\sum_{i=1}^n c_i = 1$ and $\sum_{i \in A} c_i \neq 1/2$ for every $A \subset [n]$. Let $Z_1, \ldots, Z_n$ be i.i.d. Bernoulli random variables with $\mathbb{P}(Z_i = 1) = p \geqslant 1/2$ and $\mathbb{P}(Z_i = 0) = 1 - p$, and set $Z = \sum_{i=1}^n c_i Z_i$. Prove that

$$\mathbb{P}(Z \geqslant 1/2) \geqslant p.$$

3

(i) State and prove the Four Functions theorem.

(ii) Show that if $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(n)$ then

$$|\mathcal{A} \vee \mathcal{B}| \ |\mathcal{A} \wedge \mathcal{B}| \geqslant |\mathcal{A}| \ |\mathcal{B}|.$$

(iii) For $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(n)$, write

$$\mathcal{A} - \mathcal{B} = \big\{ A \setminus B : \ A \in \mathcal{A}, \ B \in \mathcal{B} \big\}.$$

Show that

$$|\mathcal{A} - \mathcal{B}||\mathcal{B} - \mathcal{A}| \geqslant |\mathcal{A}||\mathcal{B}|.$$

4

(i) Define an independence graph of a family of events, and state the Lovász Local Lemma.

(ii) Let $k \geqslant 20$ and $s \geqslant 6k \log k$. Show that if $S \subset \mathbb{Z}$ with $|S| = s$ then $\mathbb{Z}$ has a k-colouring (depending on S) in which each colour class meets every translated copy of S.

END OF PAPER