UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 9 June, 2014 9:00 am to 11:00 am

PAPER 10

ALGEBRAIC METHODS IN INCIDENCE THEORY

*Attempt **ALL** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(A): State the Szemeredi–Trotter theorem for points and unit circles.

(B): Prove that $\Delta(P) \gtrsim |P|^{\frac{2}{3}}$ for a set P of points in the plane, where

$$\Delta(P) = \{|p-q|\colon p, q \in P\}.$$

[You may cite part (A) if you wish.]

(C): Suppose P is a set of N points in the plane. Suppose $\mathcal{L}$ is a collection of N^2 algebraic curves of degree $\leqslant \sqrt{N}$ such that for any $p, q \in P$, $p \neq q$,

$$|\{\gamma \in \mathcal{L}\colon p \in \gamma \text{ and } q \in \gamma\}| \leqslant \sqrt{N}.$$

Prove that $I(P, \mathcal{L}) \lesssim N^{\frac{5}{2}}$, where

$$I(P, \mathcal{L}) = |\{(p, \gamma) \in P \times \mathcal{L}\colon p \in \gamma\}|.$$

2

(A) Let $P \subseteq \mathbf{R}^3$ with $|P| = 12$. Write $V_d(P)$ to denote the vector space of polynomials f in $\mathbf{R}[x, y, z]$ of degree less than or equal to d such that $f(p) = 0$ for all $p \in P$. Show $\dim(V_4(P)) \geqslant 5$.

(B) Let $\mathcal{L}$ be a collection of lines in $\mathbf{R}^3$. Prove there exists a nontrivial polynomial of degree less than $C|\mathcal{L}|^{\frac{1}{2}}$ that vanishes on every line of $\mathcal{L}$. [Here C is a universal constant.]

3

Suppose a set P of n points in the plane spans fewer than $5n$ ordinary lines. Sketch a proof that P is contained in the union of C cubic curves, where C is a universal constant. [You may work in projective space if you prefer.]

END OF PAPER