UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 4 June, 2014 9:00 am to 12:00 pm

PAPER 1

COMMUTATIVE ALGEBRA

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

Throughout this paper, all rings are commutative with a 1.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define what it means for a ring R to be (i) Artinian and (ii) Noetherian.

Show that an Artinian ring is necessarily Noetherian.

Show that if R if a Noetherian ring then the power series ring $R[[X]]$ is Noetherian.

Is it true that if R is Artinian then $R[[X]]$ is Artinian?

2

Let I be a proper ideal of a Noetherian ring R.

What does it mean for a prime ideal P to be minimal over I?

What are the associated primes of the R-module R/I?

Show that the set of minimal primes over I is a non-empty subset of the set of associated primes of R/I.

Give an example where there is an associated prime of R/I which is not minimal over I.

3

Let R be a subring of a ring T.

What does it mean for T to be integral over R.

Define the dimension $\dim R$ of R and show that $\dim R = \dim T$ when T is integral over R.

Let k be a field. What is the dimension of $k[X,Y]/(XY + X^2 + Y^3)$? Justify your answer.

4

State and prove the Hilbert-Serre theorem on the rationality of Poincaré series of graded modules.

Let R be a local ring with unique maximal ideal P.

Explain how to define $d(R)$.

Give an example of a local ring R with $d(R) = 2$.

5

Define the integral closure of a subring R in a ring T.

What does it mean for an integral domain to be a valuation ring?

Let R be an integral domain with fraction field K. Show that the integral closure of R in K is the intersection of all the valuation subrings of K containing R.

6

Let P be a prime ideal of a Noetherian ring R and M be a finitely generated R-module.

Define the localisations R_P and M_P and explain how M_P is an R_P-module.

Show that M is zero if and only if M_P is zero for all prime ideals P of R.

What does it mean for M to be (i) injective and (ii) projective.

Show that M is injective if and only if M_P is injective for all prime ideals P of R.

Define the global dimension of R.

Show that if the global dimension of R is zero then all R-modules are both injective and projective.

END OF PAPER