UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 7 June, 2013 9:00 am to 12:00 pm

PAPER 9

THE KAKEYA UNIVERSE AND INCIDENCE PROBLEMS IN R^N

*Attempt **BOTH** questions from Section A,*
*and no more than **ONE** question from Section B.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

(a) State the Kakeya maximal conjecture. State the Kakeya conjecture for Minkowski dimension. Prove one implies the other.

(b) Prove the Kakeya maximal conjecture in two dimensions. You may use without proof a fact about the area of intersection of two rectangles.

2

(a) Let μ be a finite measure defined on subsets of $\mathbf{R}^2$ with the property that

$$|\widehat{d\mu}(\xi)| \lesssim \frac{1}{1+|\xi|^{1+\epsilon}}$$

for some $\epsilon > 0$ and every $\xi \in \mathbf{R}^2$. Prove

$$||\widehat{fd\mu}||_{L^2(\mathbf{R}^2)} \lesssim ||f||_{L^\infty(\mu)}$$

for every $f \in L^\infty(\mu)$.

(b) Let $\delta > 0$ be a small number, and $C \geqslant 1$ sufficiently large. Let $\psi\colon \mathbf{S}^1 \to \mathbf{R}$ be a smooth nonnegative function with $\psi(\cos(\theta), \sin(\theta)) = 1$ for $|\theta| \leqslant \frac{\delta}{C}$ and $\psi(\cos(\theta), \sin(\theta)) = 0$ for $|\theta| \geqslant \frac{2\delta}{C}$. Let σ be normalized measure on the unit circle. Prove

$$|\widehat{\psi d\sigma}(\xi)| \gtrsim \delta$$

for $\xi \in [-\frac{1}{\delta^2}, \frac{1}{\delta^2}] \times [-\frac{1}{\delta}, \frac{1}{\delta}]$.

(c) Assume

$$||\widehat{fd\sigma}||_{L^p(\mathbf{R}^2)} \lesssim ||f||_{L^p(\sigma)}$$

for a fixed p and for all $f \in L^p(\sigma)$. Suppose $\mathcal{R}$ is a collection of $\frac{1}{\delta^2} \times \frac{1}{\delta}$ rectangles whose angles with the x-axis differ by at least δ. Explain how to prove the inequality

$$\int \left(\sum_{R \in \mathcal{R}} \mathbf{1}_R \right)^{\frac{p}{2}} \lesssim \delta^{4-p} \sum_{R \in \mathcal{R}} |R|$$

with an estimate like the one in part **(b)** above.

SECTION B

3

Let $p > 10^{10}$ be a prime number and let $\mathbf{F}_p$ be the field with p elements. Suppose $N \subseteq \mathbf{F}_p^n$ has the property that for all $x \in \mathbf{F}_p^n$, there exists a line $l \ni x$ such that $\#(l \cap N) \geqslant \frac{p}{100}$. Prove $\#N \gtrsim p^n$. You may use the Schwartz–Zippel lemma without proof.

4

State and prove the joints conjecture.

END OF PAPER