UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 4 June, 2013 1:30 pm to 4:30 pm

PAPER 8

INTRODUCTION TO FOURIER ANALYSIS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Using the result that the trigonometric polynomials are uniformly dense in the continuous functions, show that if $f \in C(\mathbb{T})$ then

$$\frac{1}{2\pi}\int_{\mathbb{T}}\left|f(t)-\sum_{n=-N}^{N}\hat{f}(n)\exp(int)\right|^{2}dt \to 0$$

as $N \to \infty$.

(ii) Use (i) to establish that, if $f,\, g \in C(\mathbb{T})$ and $\hat{f}(n)=\hat{g}(n)$ for all n, then $f=g$.

(iii) Show that, if $f \in C(\mathbb{T})$ and $\sum_{n=-\infty}^{\infty}|\hat{f}(n)|$ converges then

$$\sum_{n=-N}^{N}\hat{f}(n)\exp(int) \to f(t)$$

uniformly.

(iv) Show that, if f is once continuously differentiable, then f satisfies the conditions of (iii).

(v) If $f,\, g \in C(\mathbb{T})$, show that

$$\frac{1}{2\pi}\int_{\mathbb{T}} f(t)g(t)^{*}\,dt = \sum_{n=-\infty}^{\infty}\hat{f}(n)\hat{g}(n)^{*}.$$

(vi) Give Hurewitz's proof of the classical isoperimetry inequality concerning the length and area enclosed by a curve.

(vii) In this part of the question you may assume the Fourier transform inversion theorem. Suppose that $f:\mathbb{R}\to\mathbb{R}$ is a continuous function with $\int_{-\infty}^{\infty}|f(t)|\,dt$ convergent and

$$\int_{-\infty}^{\infty} f(t)\exp(-i\lambda t)\,dt = 0$$

for $|\lambda| \geqslant \pi$. Show that

$$f(t)=\sum_{n=-\infty}^{\infty} f(n)D(t-n)$$

where D is the function given by

$$D(t)=\frac{\sin \pi t}{\pi t}$$

for $t \neq 0$ and $D(0)=1$.

2

(i) Establish the theorem of Kahane and Katznelson which states that, given any set E of zero Lebesgue measure, there exists a continuous function $f : \mathbb{T} \to \mathbb{C}$ whose partial Fourier sums $S_n(f,t)$ diverge when $t \in E$.

(ii) Show that there exists a continuous function $g : \mathbb{T} \to \mathbb{C}$ such that

$$|S_n(g,t)| \leqslant 1$$

for all $t \in \mathbb{T}$ and all $n \geqslant 0$, but $S_n(g,0)$ fails to converge.

3

(a) Develop the theory of infinite products and use it to show that, given any sequence of complex numbers $z_j \to \infty$, and any sequence of strictly positive integers $n_j \geqslant 1$ there exists an analytic function with zeros of order n_j at z_j for each j and no other zeros.

Is the result true if we omit the condition $z_j \to \infty$? Give reasons.

Suppose the conditions of the first paragraph apply and f and g are analytic functions with zeros of order n_j at z_j for each j and no other zeros. Show that there exists an analytic function h with $f(z) = e^{h(z)}g(z)$. If k is an analytic function with $f(z) = e^{k(z)}g(z)$, what is the relation between h and k? Give reasons.

(b) Starting from first principles, establish the Fourier inversion formula

$$f(x) = A \sum_{\chi \in \hat{G}} \hat{f}(\chi)\chi(x)$$

for finite Abelian groups. (You are not asked to find the constant A.)

[Part (b) carries roughly half the weight of part (a).]

4

(a) Show that there exist strictly positive numbers A and B such that (if n is large) the number $N(n)$ of primes between 1 and n satisfies the condition

$$A\frac{n}{\log n} \leqslant N(n) \leqslant B\frac{n}{\log n}.$$

(b) Let Ω is an open set in $\mathbb{C}$ with $\Omega \supseteq \{z : \Re z \geqslant 0\}$. Suppose that $F : \Omega \to \mathbb{C}$ is an analytic function and $f : [0, \infty) \to \mathbb{R}$ a bounded locally integrable function such that

$$F(z) = \int_0^\infty f(t)e^{-tz}\,dz$$

for $\Re z > 0$. Show that $\displaystyle\int_0^\infty f(t)\,dt$ converges.

Give an example to show that the result may fail if we only demand

$$\Omega \supseteq \{z : \Re z > 0\}.$$

[*The two parts of this question are of roughly equal weight.*]

END OF PAPER