UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 31 May, 2013 1:30 pm to 4:30 pm

PAPER 75

QUANTUM CONDENSED MATTER FIELD THEORY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The transverse field quantum Ising model is defined by the Hamiltonian

$$\hat{H} = -4J \sum_{n=1}^{N} \left[\hat{S}_n^x \hat{S}_{n+1}^x + \frac{g}{2} \hat{S}_n^z \right] ,$$

where $\hat{\mathbf{S}}_n$ denotes the quantum spin 1/2 operator at lattice site n, and the boundary conditions are periodic such that $\hat{\mathbf{S}}_{n+N} \equiv \hat{\mathbf{S}}_n$.

(a) Taking the spin exchange constant $J > 0$, construct the ground state wavefunctions and energies of the Hamiltonian for transverse field strength $g = 0$ and $g \to \infty$, and comment on their degeneracy.

(b) In one dimension, the operator algebra of spin 1/2 can be generated by spinless fermion operators through the Jordan-Wigner representation,

$$\hat{S}_n^+ = \exp\left[i\pi \sum_{m=1}^{n-1} c_m^\dagger c_m \right] c_n, \qquad \hat{S}_n^- = (\hat{S}_n^+)^\dagger, \qquad \hat{S}_n^z = \frac{1}{2} - c_n^\dagger c_n ,$$

where the Fermion operators, c_m obey anticommutation relations, $[c_m, c_n^\dagger]_+ = \delta_{mn}$. Neglecting boundary-like terms (which are small in the limit $N \to \infty$), show that the Hamiltonian can be brought to the form

$$\hat{H} = J \sum_k \begin{pmatrix} c_k^\dagger & ic_{-k} \end{pmatrix} \begin{pmatrix} (g - \cos k) & -\sin k \\ -\sin k & -(g - \cos k) \end{pmatrix} \begin{pmatrix} c_k \\ -ic_{-k}^\dagger \end{pmatrix} ,$$

where the sum runs over discrete Fourier modes, k.

(c) Obtain an expression for the ground state energy of the Hamiltonian, and show that the spectrum of quasi-particle excitations is given by $\epsilon_k = 2J(1 + g^2 - 2g\cos k)^{1/2}$. Sketch and comment on the form of ϵ_k for (i) $g = 0$, (ii) $g \to \infty$, and (iii) $g = 1$.

(d) Show that the ground-state expectation value of $\hat{S}_n^z$ is given by the sum,

$$\sum_{n=1}^{N} \langle \text{g.s.} | \hat{S}_n^z | \text{g.s.} \rangle = \sum_k \frac{J(g - \cos k)}{\epsilon_k} .$$

2

The quantum spin S Heisenberg antiferromagnet is described by the Hamiltonian,

$$\hat{H} = J \sum_{\langle ij \rangle} \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j,$$

where $\hat{\mathbf{S}}_i$ denotes the spin S operator at lattice site i, the exchange constant $J > 0$, and the sum runs over nearest neighbour sites i and j of a regular d-dimensional lattice.

(a) Define the nature of the *classical* ground states for a general bipartite lattice and comment on their form for one example of a non-bipartite lattice.

(b) Setting $\hbar = 1$, show that the transformation,

$$\hat{S}^z = S - a^\dagger a, \qquad \hat{S}^- = (\hat{S}^+)^\dagger = (2S)^{1/2} a^\dagger \left(1 - \frac{a^\dagger a}{2S}\right)^{1/2},$$

where operators a and $a^\dagger$ obey Bose commutation relations, $[a, a^\dagger]_- = 1$, is consistent with quantum spin algebra.

(c) Focusing on the one-dimensional quantum Heisenberg antiferromagnetic spin chain, with periodic boundary conditions, $\hat{\mathbf{S}}_{m+N} = \hat{\mathbf{S}}_m$, use this transformation to show that

$$\hat{H} = -NJS(S+1) + JS \sum_k \begin{pmatrix} a_k^\dagger & a_{-k} \end{pmatrix} \begin{pmatrix} 1 & \gamma_k \\ \gamma_k & 1 \end{pmatrix} \begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix} + O(S^0),$$

where $\gamma_k = \cos k$.

(d) By implementing an appropriate transformation, show the Hamiltonian can be brought to the diagonal form

$$\hat{H} = -NJS(S+1) + \sum_k \omega_k \left[\alpha_k^\dagger \alpha_k + \frac{1}{2}\right] + O(S^0),$$

where the operators $\alpha_k^\dagger$ also obey Bose commutation relations. Comment on the form of the dispersion, ω_k. Explain how this result generalizes to the d-dimensional hypercubic lattice.

(e) By considering the sublattice magnetization of the ground state in the one-dimensional system,

$$\langle \text{g.s.} | \frac{1}{N} \sum_m (-1)^m \hat{S}_m^z | \text{g.s.} \rangle,$$

comment on the nature of the spin order. Generalizing this result, comment on what happens in higher dimension.

3

A quantum particle moves in a one-dimensional double well potential $V(q) = \frac{1}{(8a^2)} m\omega^2 (q^2 - a^2)^2$.

(a) In the Euclidean time formulation, obtain an expression for the quantum transition amplitudes $\langle a | e^{-\hat{H}\tau/\hbar} | \pm a \rangle$ in the form a Feynman path integral.

(b) Taking $\omega\tau \gg 1$, show that the transition amplitudes can be written in the form,

$$\langle a | e^{-\hat{H}\tau/\hbar} | \pm a \rangle \approx \sum_{n \text{ even / odd}} \frac{e^{-\omega\tau/2}}{n!} \left(\tau K e^{-S_0/\hbar} \right)^n .$$

Obtain an expression for S_0 and comment on the meaning of K. Comment on the validity of the approximation. Performing the sum on n, discuss the physical interpretation of the result.

For the quantum haromic oscillator potential $V(q) = \frac{1}{2} m\omega^2 q^2$, you may note that the following identity for the transition amplitude,

$$\langle 0 | e^{-\hat{H}\tau/\hbar} | 0 \rangle = \left(\frac{m\omega}{2\pi\hbar \sinh(\omega\tau)} \right)^{1/2} .$$

(c) Suppose now that the quantum particle moves in a periodic lattice potential V with periodicity a and minima at na, with n integer. Making use of the result above, obtain an expression for the Feynman amplitude $\langle na | e^{-\hat{H}\tau/\hbar} | ma \rangle$, with m and n both integer. Then, making use of the identity $\delta_{qq'} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i(q-q')\theta}$, show that

$$\langle na | e^{-\hat{H}\tau/\hbar} | ma \rangle \sim e^{\omega\tau/2} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-i(n-m)\theta} \exp\left[\frac{2\Delta\epsilon\, \tau}{\hbar} \cos\theta \right] .$$

Explain the physical meaning of $\Delta\epsilon$.

(d) Show that this expression is consistent with the low energy particle spectrum $\epsilon_p = \hbar\omega/2 - 2\Delta\epsilon \cos(pa)$. Explain why.

4

A gas of bosonic particles of mass m, confined to a volume L^d, and subject to a local repulsive contact interaction of strength g, is described by the d-dimensional quantum Hamiltonian,

$$\hat{H} = \int_0^L \mathrm{d}^d r \left[a^\dagger(\mathbf{r}) \frac{\hat{\mathbf{p}}^2}{2m} a(\mathbf{r}) + \frac{g}{2} a^\dagger(\mathbf{r})\, a^\dagger(\mathbf{r})\, a(\mathbf{r})\, a(\mathbf{r}) \right] ,$$

where the boson operators $a^\dagger(\mathbf{r})$ and $a(\mathbf{r})$, respectively, create and annihilate particles at position $\mathbf{r}$.

(a) Without detailed derivation, show that the partition function, $\mathcal{Z} = \mathrm{tr}\, e^{-\beta(\hat{H}-\mu\hat{N})}$, can be cast as a function field integral,

$$\mathcal{Z} = \int D(\bar{\psi}, \psi)\, \mathrm{e}^{-S[\bar{\psi},\psi]} ,$$

where $\bar{\psi}$ and ψ denote complex fields. Specify both the action, $S[\bar{\psi}, \psi]$, and state the boundary conditions on the fields $\bar{\psi}$ and ψ.

(b) Focussing on the low-temperature system, show that, in the mean-field (saddle-point) approximation, the ground state forms a Bose-Einstein condensate. Identify the continuous symmetry that is broken in the ground state, describe the corresponding manifold of degeneracy associated with possible ground states. What are the physical consequences of symmetry breaking?

(c) Parameterising the complex fields as $\psi = \sqrt{\rho}\, e^{i\phi}$, where $\sqrt{\rho}$ and ϕ are real, show that the action involves a set of massive and massless fluctuations.

(d) Expanding the action to leading order in fluctuations of ρ and ϕ around the saddle-point, and discarding gradient terms associated with ρ, show that the integral over the massive fluctuations leads to a field integral of the form, $\mathcal{Z} = \mathrm{e}^{-S_0} \int D\phi\, \mathrm{e}^{-S_{\mathrm{eff}}[\phi]}$, where S_0 is constant, and

$$S_{\mathrm{eff}}[\phi] = \frac{1}{2} \int_0^\beta \mathrm{d}\tau \int_0^L \mathrm{d}^d r \left[\frac{1}{g} (\partial_\tau \phi)^2 + \frac{\rho_0}{m} (\partial \phi)^2 \right] .$$

Without detailed derivation, by identifying the relation of the effective action, S_{eff}, to that of the quantum harmonic chain, or otherwise, obtain the spectrum of low-energy excitations of the weakly interacting Bose gas.

END OF PAPER