UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 11 June, 2013 9:00 am to 12:00 pm

PAPER 74

TOPOS THEORY

*Attempt no more than **THREE** questions, of which at most **TWO** should be from Section A.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

Starting from the definition of a topos, show that if $\mathcal{E}$ is a topos then the power-object functor $P : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ is monadic. Deduce that a logical functor which has a left adjoint also has a right adjoint. Give examples (with brief justification) of functors F and G between toposes such that both have left adjoints, F preserves exponentials and G preserves the subobject classifier, but neither has a right adjoint.

[Standard theorems on monadicity, and on the lifting of adjoints, may be assumed.]

2

Show that if $\mathbb{G}$ is a cartesian comonad on a topos, then the category of $\mathbb{G}$-coalgebras is a topos. Hence show that any geometric morphism between toposes can be factored as one whose inverse image is faithful, followed by one whose direct image is full and faithful.

3

Explain carefully what is meant by a cartesian (first-order) theory. Sketch the construction of the syntactic category $\mathcal{C}_{\mathbb{T}}$ associated with a cartesian theory $\mathbb{T}$, and show that cartesian functors $\mathcal{C}_{\mathbb{T}} \to \mathcal{E}$ correspond to $\mathbb{T}$-models in $\mathcal{E}$. By considering representable functors, or otherwise, deduce the classical completeness theorem for cartesian theories: if σ is a cartesian sequent relative to $\mathbb{T}$, which is satisfied in all $\mathbb{T}$-models in **Set**, then σ is derivable in $\mathbb{T}$.

SECTION B

4

Recall that an object A of a coherent category is said to be *decidable* if the diagonal $A \rightarrowtail A \times A$ is a complemented subobject. Show that an object F of a functor category $[\mathcal{C}, \mathbf{Set}]$ is decidable iff $F(f)$ is injective for every morphism f of $\mathcal{C}$.

If M is a monoid and A and B are (left) M-sets, show that the exponential B^A in $[M, \mathbf{Set}]$ may be identified with the set of M-equivariant maps $f \colon M \times A \to B$, with M-action given by

$$(m.f)(m', a) = f(m'm, a) \ .$$

Hence show that

(a) if M satisfies the condition

$$(\forall m \in M)(\exists p, q \in M)(pmq = p)$$

then B^A is decidable whenever B is. [*Hint: First show that if* $f, g \in B^A$ *satisfy* $f \neq g$, *then* $f(1, a) \neq g(1, a)$ *for some* $a \in A$.]

(b) if M is the free monoid on two generators x and y, A is the set of natural numbers with M-action given by $x.n = y.n = n + 1$ for all n, and $B = \{0, 1\}$ with trivial M-action, then B^A is not decidable. [*Hint: Consider the function* f *defined by* $f(w, n) = 1$ *if* w *is a word of length* $> n$ *ending with* x, $f(w, n) = 0$ *otherwise.*]

5

Explain how the subobject classifier in a topos $\mathcal{E}$ is given the structure of a Heyting algebra. Define the quasi-closed local operator $q(U)$ associated with a subterminal object U of $\mathcal{E}$, and show that a subtopos $\mathbf{sh}_j(\mathcal{E})$ of $\mathcal{E}$ is Boolean if j is quasi-closed.

Show that the composite geometric morphism

$$\mathbf{sh}_{q(\top)}(\mathcal{E}/\Omega) \longrightarrow \mathcal{E}/\Omega \longrightarrow \mathcal{E} \ ,$$

where $\top$ is the generic subobject in $\mathcal{E}$ regarded as a subterminal object of $\mathcal{E}/\Omega$, is surjective. Explain how this result may be used to derive a 'classical completeness theorem' for coherent theories.

6

Explain what is meant by the *regular coverage* R on a small regular category $\mathcal{C}$, and explain briefly why it is subcanonical (that is, the representable functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ are sheaves).

If F' is a subfunctor of an R-sheaf $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, show that the closure of $F' \rightarrowtail F$ (for the local operator on $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ corresponding to R) is the functor F'', where

$$F''(A) = \{x \in F(A) \mid F(\alpha)(x) \in F'(B) \text{ for some cover } \alpha\colon B \twoheadrightarrow A\} \ .$$

[*Hint: First show that* F'' *is a sheaf.*]

Deduce that the representable functors are irreducible as objects of $\mathbf{Sh}(\mathcal{C}, R)$, in the sense that if $\mathcal{C}(-, A)$ is the union of a family of subsheaves $(F_i \mid i \in I)$, then some F_i must be the whole of $\mathcal{C}(-, A)$.

By applying this result to the classifying topos of a regular theory $\mathbb{T}$, deduce that if $\vec{x}.\phi$ and $\vec{x}.\psi_i$ $(1 \leqslant i \leqslant n)$ are regular formulae-in-context over the signature of $\mathbb{T}$, and the coherent sequent $(\phi \vdash_{\vec{x}} \bigvee_{i=1}^{n} \psi_i)$ is derivable in $\mathbb{T}$, then $(\phi \vdash_{\vec{x}} \psi_i)$ is derivable for some i.

[You may assume that the classifying topos of $\mathbb{T}$ has the form $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, R)$ where $\mathcal{C}_{\mathbb{T}}$ is (the regular version of) the syntactic category of $\mathbb{T}$.]

END OF PAPER