UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 6 June, 2013 1:30 pm to 4:30 pm

PAPER 73

FLUID DYNAMICS OF CLIMATE

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider the motion of an incompressible Boussinesq fluid rotating about a vertical axis with angular velocity $\frac{1}{2}f$ and with constant buoyancy frequency N. Show that for small amplitude motions the pressure satisfies the equation

$$\nabla^2 p_{tt} + f^2 p_{zz} + N^2 {\nabla_H}^2 p \quad = \quad Q(\boldsymbol{x}),$$

for arbitrary Q. Show further that the potential vorticity of the motion

$$q \equiv \zeta - f(b/N^2)_z,$$

is related to Q by

$$Q \quad = \quad \rho_0 f N^2 q.$$

Consider plane-waves independent of the y-direction. Then

(i) Show by direct calculation using the dispersion relation that the waves have zero PV.

(ii) Calculate the phase and group velocities and show that they are perpendicular, and that their vertical components are in opposite directions.

(iii) Calculate the kinetic and potential energy of the waves and show that equipartition holds in the rotating case.

2

A homogeneous, incompressible fluid of uniform depth H is at rest on an f-plane. State the assumptions behind the 'shallow water approximation' and show that this implies that the pressure is hydrostatic.

Write down the shallow water equations for small amplitude motions with free surface elevation η. Show that these equations imply that the shallow water perturbation potential vorticity

$$\frac{\zeta}{f} - \frac{\eta}{H},$$

where η is the relative vorticity, is conserved.

Consider a coastal current formed by an elevation of the free surface in $x < 0$ and bounded by a vertical 'coast' at $x = 0$. The initial free surface elevation is

$$\eta \quad = \quad \begin{cases} \eta_0 & -L < x < 0 \\ 0 & x < -L. \end{cases}$$

Using conservation of perturbation potential vorticity, conservation of mass and continuity of η and v, show that the final state is

$$\eta(x) \quad = \quad \eta_0 e^{\frac{x}{a}} \sinh \frac{L}{a} + \begin{cases} \eta_0 \left(1 - \cosh \frac{L+x}{a}\right) & -L < x < 0 \\ 0 & x < -L, \end{cases}$$

where $a = \sqrt{gH}/f$. Find the flow in the adjusted state and show that the velocity satisfies the *no-slip* condition at the coast. Discuss the final shape of the free surface in the limits $L >> a$ and $L << a$.

3

Consider the following steady, linearised momentum equation describing the large-scale circulation of the ocean,

$$f\hat{k} \times \mathbf{u}_h = -\frac{1}{\rho_0}\nabla_h p + \nu\nabla^2\mathbf{u}_h, \tag{1}$$

where ρ_0 is a constant density under the Boussinesq approximation, $\mathbf{u}_h = (u, v, 0)$ is the horizontal velocity, $\nabla_h \equiv (\partial_x, \partial_y, 0)$, f is the vertical component of the Coriolis parameter, and ν is the kinematic viscosity. In this coordinate frame, the unit vector in the x-direction points to the east, the unit vector in the y-direction points to the north, and $z = 0$ and $z = -H$ correspond to the surface and bottom of the ocean respectively. Assume that H is constant.

(i) Assuming that the horizontal scales of motion are much larger than the ocean depth, and neglecting bottom stress, derive the equation for the depth-integrated *Sverdrup flow*, in terms of the surface wind stress

$$\boldsymbol{\tau}_w \equiv \rho_0\nu \left.\frac{\partial \mathbf{u}_h}{\partial z}\right|_{z=0}. \tag{2}$$

(ii) Frictional effects at the seafloor are often modeled using a linear bottom stress of the form

$$\boldsymbol{\tau}_b \equiv \rho_0\nu \left.\frac{\partial \mathbf{u}_h}{\partial z}\right|_{z=-H} = \gamma\mathbf{u}_h. \tag{3}$$

Using this model for the bottom stress, derive an equation for the streamfunction associated with the depth-integrated flow, $\overline{\psi}$. Consider a rectangular basin with $0 < x < 1$ and $0 < y < 1$, with no-slip boundary conditions. Let $W \equiv \nabla_h \times \boldsymbol{\tau}^w$ be the wind stress curl, and assume that $W(y)$ and its derivatives vanish at the north/south boundaries, $y = 0, 1$. Assuming that ν and γ are small parameters and using a boundary layer method, write down an ordinary differential equation for the streamfunction in the eastern and western boundary layers.

(iii) Identify two limiting cases based on the sizes of β, ν, and γ, and clearly state when each limit is valid. In each of these limits, find the form of the general solution valid throughout the domain. Based on the solution form, what is the expected thickness of the eastern and western boundary layers in each limit. [*Hint: You don't need to solve for the constants in the general solution.*]

4

Start from the two-layer shallow water equations:

$$\frac{D_1\mathbf{u}_1}{Dt} + f\hat{\mathbf{k}} \times \mathbf{u}_1 = -g\nabla(h_1 + h_2), \tag{1}$$

$$\frac{D_2\mathbf{u}_2}{Dt} + f\hat{\mathbf{k}} \times \mathbf{u}_2 = -g\nabla(h_1 + h_2) + g'\nabla h_1, \tag{2}$$

$$\frac{\partial h_i}{\partial t} + \nabla \cdot (h_i\mathbf{u}_i) = 0, \quad \text{for} \quad i = 1, 2, \tag{3}$$

where $D_i/Dt = \partial/\partial t + \mathbf{u}_i \cdot \nabla$ is the material derivative within each layer, h_i is the depth of each layer, f is the vertical component of the Coriolis parameter, and $g' = (\rho_2 - \rho_1)g/\rho_2$ is the reduced gravity.

(i) Neglecting the free surface displacement and assuming a flat bottom, derive the two-layer quasi-geostrophic equations. Clearly state any other approximations needed.

(ii) Derive the QG energy equation by multiplying the QG equation in each layer by $H_i\psi_i$ with $i = 1, 2$, where H_i is the constant depth of each layer in a state of rest, and ψ_i is the streamfunction.

(iii) Integrating over a horizontal domain where lateral boundary fluxes vanish, show that the QG energy satisfies a conservation equation of the form

$$\frac{\partial}{\partial t}\int\int \text{QE } dxdy = 0, \tag{4}$$

and give an expression for QE in the integrand. Identify the terms in QE corresponding to kinetic and potential energy, then show that the ratio of the potential energy to the baroclinic kinetic energy scales with

$$\frac{l^2 f_0^2}{g' H_i}, \tag{5}$$

where l is a characteristic horizontal scale associated with the flow, and $\widetilde{\psi} \equiv (\psi_1 - \psi_2)/2$ is the streamfunction associated with the baroclinic velocity. Comment on the relative importance of kinetic and potential energy to large and small scale dynamics.

END OF PAPER