MATHEMATICAL TRIPOS Part III

Monday, 10 June, 2013 1:30 pm to 3:30 pm

PAPER 71

SOLIDIFICATION OF FLUIDS

There are ***THREE*** *questions in total.*

You may attempt any number of questions.

Full marks can be gained from complete answers to ***TWO*** *questions.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A deep layer of fresh water at temperature $T_m = 0°\text{C}$ is placed above and in contact with a deep layer of salty water of temperature $T_\infty < 0°\text{C}$ and salinity C_0 less than the eutectic concentration. A layer of ice forms between the two fluid layers.

Describe the subsequent evolution of the system. You should include a diagram of the physical system showing the temperature and salinity fields, a phase diagram showing the trajectory of temperature and salinity, a calculation of the positions of both phase boundaries, and an explanation of the physics causing phase change.

You should assume that $\epsilon \ll \mathcal{S}^{-1} \ll 1$, where the Stefan number $\mathcal{S} \equiv L/c_p(T_m - T_\infty)$, L is the latent heat of fusion, c_p is the specific heat capacity and $\epsilon \equiv \sqrt{D/\kappa}$, where D and κ are the diffusivities of salt and heat respectively.

You may make any additional assumptions you wish but these must be stated clearly and justified. All symbols that you introduce must be clearly defined.

2

A supercooled pure melt of freezing temperature T_m and temperature $T_\infty < T_m$ solidifies at constant speed V to form a solid of equal thermo-physical properties. The thickness of the liquid and solid layers are both much greater than κ/V, where κ is their thermal diffusivity. Assuming that the solid–liquid interface is at temperature T_m, show that the Stefan number $\mathcal{S} = L/c_p\Delta T$ must be equal to unity, where L is the latent heat of fusion, c_p is the specific heat capacity and $\Delta T = T_m - T_\infty$.

Show that perturbations to the solid–liquid interface of wavenumber $\alpha \gg V/\kappa$ have growth rate σ given by

$$\frac{\kappa}{V^2}\sigma = -1 + \frac{\kappa}{V}\alpha - \frac{2\gamma T_m}{\rho L \Delta T}\frac{\kappa^2}{V^2}\alpha^3,$$

where ρ is the density of the freezing material and γ is the solid–liquid surface energy.

Show that there is morphological instability if

$$\frac{\kappa}{V}\frac{\rho L \Delta T}{\gamma T_m} > \frac{27}{2}.$$

Use your analysis to explain qualitatively how you would expect a snowflake to develop.

3

A horizontal layer of ice of thickness $h(t)$ sits on top of a rigid, porous, highly conductive sheet of thickness d and permeability Π that separates it from a bath of water held in a liquid state at a uniform temperature $T_m = 0°\mathrm{C}$ by a small heating element. There is no phase change at the top of the ice, which is held at a fixed temperature $T_h < T_m$ as the thickness of the ice increases with time t. The pressure at the top of the layer of ice is the same as the pressure in the water bath. Van der Waals interactions between the ice and the porous sheet create a disjoining force per unit area $p_T = A/6\pi a^3$ across a pre-melted liquid film of water of thickness $a \ll h$ separating the ice from the sheet.

Assume that the temperature field across the ice is quasi-steady and show that the thickness of the ice evolves according to

$$\frac{dh}{dt} = \frac{\Pi}{\mu d}\left[\left(\frac{\rho L \Delta T}{T_m}\right)^{3/4}\left(\frac{A}{6\pi h^3}\right)^{1/4} - \rho g h\right],$$

where $\Delta T = T_m - T_h$, μ is the dynamic viscosity of water, g is the acceleration due to gravity and it is assumed that ice and water have the same density ρ.

In suitable dimensional variables, which should be defined, the differential equation can be written as

$$\frac{d\eta}{d\tau} = \eta^{-3/4} - \eta.$$

Draw a sketch of $\eta(\tau)$, showing explicitly how η behaves at small and large times, and find the explicit solution for $\eta(\tau)$.

END OF PAPER