UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 10 June, 2013 9:00 am to 11:00 am

PAPER 70

SOUND GENERATION AND PROPAGATION

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Starting from the wave equation

$$\left(\frac{\partial^2}{\partial t^2} - {c_0}^2\nabla^2\right)(\rho - \rho_0) = -\frac{\partial}{\partial x_i}\big(F_i\delta(f)|\boldsymbol{\nabla} f|\big),$$

where $f = f(\boldsymbol{x}, t)$ and $F_i = F_i(\boldsymbol{x}, f)$, derive the solution in integral form

$$\rho(\boldsymbol{x}, t) - \rho_0 = -\frac{\partial}{\partial x_i}\iint\left[\frac{F_i(\boldsymbol{y})h_p h_q}{4\pi {c_0}^2|\boldsymbol{x}-\boldsymbol{y}|\,|1-M_r|}\right]_{\tau=\tau^*}\mathrm{d}p\mathrm{d}q \qquad (*)$$

where M_r and τ^* should be specified, the surface $f(\boldsymbol{y}, \tau) = 0$ is given by $\boldsymbol{y} = \boldsymbol{y}(p, q, \tau)$ with p and q orthogonal coordinates, and h_p and h_q are $|\partial\boldsymbol{y}/\partial p|$ and $|\partial\boldsymbol{y}/\partial q|$ respectively. You may use, without proof, the identity

$$\int_{\mathbb{R}^3} A\delta(f)|\boldsymbol{\nabla} f|\,\mathrm{d}^3\boldsymbol{y} = \iint A h_p h_q\,\mathrm{d}p\mathrm{d}q.$$

(b) The interior of a *rigid* object *moving with the fluid* is given by $f < 0$, and hence the sound generated by the object may be approximated by (*), with $\boldsymbol{F}$ being the force exerted by the object on the fluid. Explain briefly why the other two terms of the Ffowcs Williams–Hawkings equation may be neglected in this case, stating any other assumptions needed. How does (*) simplify for a compact object heard in the far field?

A wind turbine has two *thin* blades of length a that rotate *slowly* in the y, z-plane with angular frequency Ω. The blades are horizontal (in the $\boldsymbol{e_y}$ direction) at time $t = 0$. At a distance r from the centre of rotation, the force each blade exerts on the fluid (integrated over the blade cross-section) is $\boldsymbol{F} = (2Fr/a^2)\boldsymbol{e_\phi} + (2Dr/a^2)\boldsymbol{e_x}$, where $\boldsymbol{e_\phi}$ is the direction of motion of the blade. An observer stands in the far field at $\boldsymbol{x} = R(\cos\theta\,\boldsymbol{e_x} + \sin\theta\,\boldsymbol{e_y})$ with $R \gg c_0/\Omega$.

(c) Why does the compact limit imply $\Omega a/c_0 \ll 1$? What is $|1 - M_r|$ in this limit? What is the sound radiated from a single blade in the compact limit? What does the compact limit result give for the total sound from both blades?

(d) In the limit $\Omega a/c_0 \ll 1$, show that

$$\Omega\tau^* = \Omega\tau_0^* + \frac{\Omega r}{c_0}\sin\theta\cos(\Omega\tau_0^*) + O\big(1/R, (\Omega a/c_0)^2\big),$$

where $\tau_0^* = t - R/c_0$. What are the first two nonzero terms of $\boldsymbol{F}$ in this limit? What are the first two nonzero terms of $|1 - M_r|^{-1}$? Using (*), calculate the first nonzero term of the sound radiated from both blades to the far field. Explain briefly on physical grounds the directivity and time-dependence of your solution.

2

In the absence of sound, a stationary fluid of density ρ_0, pressure p_0 and sound speed c_0 occupies the region $y > 0$. The horizontal surface $y = 0$ reacts with an impedance $Z = P/V$, where a pressure $p = p_0 + P\exp\{\mathrm{i}\omega t - \mathrm{i}kx\}$ gives a surface velocity in the $+y$-direction of $-V\exp\{\mathrm{i}\omega t - \mathrm{i}kx\}$. Throughout this question, you may like to consider $\mathrm{Im}(\omega) < 0$.

(a) An incoming plane wave of amplitude A and frequency ω propagates towards the surface at an angle θ to the horizontal. Show that surface reflects a plane wave of amplitude R given by

$$\frac{A+R}{A-R} = \frac{Z\sin\theta}{\rho_0 c_0}.$$

Comment *briefly* on the cases (i) $Z \to 0$, (ii) $Z \to \infty$, (iii) $R/A \to 0$, and (iv) $R/A \to \infty$, giving a physical description of both the behaviour of the surface and the behaviour of the reflected wave in each case.

(b) Suppose the surface at $y = 0$ consists of a thin elastic sheet of mass-per-unit-area m stretched along $y = 0$ with tension T, with a stationary fluid of density ρ_0, pressure p_0 and sound speed c_0 occupying $y < 0$. For small displacements, the displacement η of the sheet in the y-direction is governed by

$$m\frac{\partial^2\eta}{\partial t^2} = T\frac{\partial^2\eta}{\partial x^2} - p_+ + p_-, \qquad (\dagger)$$

where p_+ and p_- are the pressures in the fluids just above and just below the sheet respectively. What impedance $Z(k,\omega)$ does the fluid in $y > 0$ see? How do the limits (i) $m \to \infty$, (ii) $T \to \infty$ and (iii) $m = T = 0$ correspond to your physical answers to part (a)? *[Hint: $T \to \infty$ is nearly, but not quite, the same as $m \to \infty$.]*

(c) In the absence of *incoming* sound in the fluids, what is the dispersion relation $D(k,\omega)$ for waves propagating along the elastic sheet (†)? For values of k and ω satisfying $D(k,\omega) = 0$, what is the value of $Z(k,\omega)$ calculated in part (b)? In light of this, how should one interpret the limit $R/A \to \infty$ in part (a)?

(d) Now suppose sound is generated by an oscillating point force acting on the elastic sheet, so that (†) becomes

$$m\frac{\partial^2\eta}{\partial t^2} = T\frac{\partial^2\eta}{\partial x^2} - p_+ + p_- + F\delta(x)e^{\mathrm{i}\omega t}.$$

By Fourier transforming in x, show that the sound generated by this forcing in the far field for $y > 0$ is, provided no poles contribute,

$$\rho - \rho_0 \sim -\sqrt{\frac{\omega}{2\pi r}}\,\frac{F\rho_0\exp\{\mathrm{i}\omega(t - r/c_0) + \mathrm{i}\pi/4\}\sin\theta}{\left(\cos^2\theta - m{c_0}^2/T\right)\mathrm{i}\omega T\sin\theta - 2\rho_0 c_0}.$$

Explain *briefly*, without further calculation (but possibly using a sketch), how you would decide when the poles contribute. What do these poles represent physically?

[*Hint: You may assume without proof that as $r \to \infty$*

$$\int_{\mathcal{C}_{\mathrm{SD}}} f(k)e^{-r(\mathrm{i}k\cos\theta + \gamma\sin\theta)}\,\mathrm{d}k \sim \sqrt{2\pi k_0/r}\,f(k_s)e^{\mathrm{i}\pi/4 - \mathrm{i}k_0 r}\sin\theta,$$

where $\mathcal{C}_{\mathrm{SD}}$ is the steepest descent contour, $k_0 = \omega/c_0$, $\gamma^2 = k^2 - {k_0}^2$, and the dominant contribution comes from the neighbourhood of $k_s = k_0 \cos\theta$.]

3

Burgers' equation is

$$\frac{\partial f}{\partial z} - f\frac{\partial f}{\partial \theta} = \varepsilon\frac{\partial^2 f}{\partial \theta^2}.$$

The inviscid Burgers' equation is obtained by setting $\varepsilon = 0$.

(a) Show that the inviscid Burgers' equation with initial conditions $f(0,\theta) = f_0(\theta)$ has solution $f\big(z, \theta_0 - f_0(\theta_0)z\big) = f_0(\theta_0)$. Show also that if there is a weak shock at $\theta_s(z)$ then

$$\frac{\mathrm{d}\theta_s}{\mathrm{d}z} = -\tfrac{1}{2}\lim_{\delta\to 0}\Big(f(z,\theta_s+\delta) + f(z,\theta_s-\delta)\Big).$$

Solve the inviscid Burgers' equation for the initial conditions

$$f(0,\theta) = \begin{cases} 0 & \theta < 0 \\ U & \theta > 0\,, \end{cases} \qquad (+)$$

being careful to distinguish between $U < 0$ and $U > 0$.

[Hint: it may help to sketch the characteristics first. For $U < 0$, think of $f(0,\theta)$ as being continuous but very steep at $\theta = 0$.]

(b) For $\varepsilon \neq 0$, show that the Cole–Hopf transformation

$$f = 2\varepsilon\frac{\partial}{\partial \theta}\log\psi$$

can be used to solve Burgers' equation when ψ satisfies a diffusion equation. Given that the general solution to the diffusion equation is

$$\psi(z,\theta) = \frac{1}{\sqrt{4\pi\varepsilon z}}\int_{-\infty}^{\infty}\psi(0,\phi)\exp\left\{-\frac{(\phi-\theta)^2}{4\varepsilon z}\right\}\,\mathrm{d}\phi,$$

show that the solution to the full Burgers' equation for the initial conditions given in (+) is

$$f(z,\theta) = \frac{U}{1+\alpha\exp\big\{-U(2\theta+Uz)/4\varepsilon\big\}},$$

where

$$\alpha = \frac{\displaystyle\int_{\theta}^{\infty}\exp\big\{-y^2/4\varepsilon z\big\}\,\mathrm{d}y}{\displaystyle\int_{-(\theta+Uz)}^{\infty}\exp\big\{-y^2/4\varepsilon z\big\}\,\mathrm{d}y}.$$

Being careful about the sign of U, what happens (i) as $\theta\to\infty$, (ii) as $\theta\to-\infty$, and (iii) when $\alpha = 1$? How does this compare with your inviscid solution found in (a)?

$$\left[\textit{Hint:}\qquad \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty}e^{-t^2}\,\mathrm{d}t \quad\sim\quad \frac{e^{-x^2}}{x\sqrt{\pi}} \quad \textit{as}\quad x\to+\infty.\right]$$

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