UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 6 June, 2013 1:30 pm to 4:30 pm

PAPER 7

ASPECTS OF ANALYSIS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

What is a separated dual pair $\langle E, F\rangle$ of real vector spaces? What is the weak topology $\sigma(E,F)$ on E? Determine the space of $\sigma(E,F)$-continuous linear functionals on E.

Suppose now that $(E, \|.\|)$ is a separable infinite-dimensional real Banach space, with dual space E', and that B' is the unit ball of E'. Give E' the weak* topology $\sigma(E', E)$. Which of the following statements are always true? Justify your answers.

(i) B' is metrizable.

(ii) B' is separable.

(iii) E' is separable.

(iv) E' is not metrizable.

2

Show that a weakly compact convex subset K of a real Banach space $(E, \|.\|)$ is the closed convex cover of the set $Ex(K)$ of its extreme points.

What are the extreme points of the closed unit ball B of the space $(C(C), \|.\|_\infty)$ of continuous real-valued functions on the Cantor set C? Show that B is the closed convex cover of the set $Ex(B)$ of its extreme points. By considering the sequence $(f_n)_{n=1}^\infty$ of functions, where $f_n(x) = \min(3^n x, 1)$ for $x \in C$, or otherwise, show that B is not weakly compact.

3

State and prove Wiener's Lemma.

Suppose that V is an open subset of $\mathbf{R}^d$ of finite Lebesgue measure. Show that there exists a disjoint sequence (U_n) of open balls contained in V such that $\lambda_d(V \setminus (\cup_n U_n)) = 0$.

Suppose that μ is a Borel probability measure on $\mathbf{R}^d$. If $x \in \mathbf{R}^d$ and $r > 0$, let $U_r(x) = \{y : \|x-y\| < r\}$, let $A_r(x) = \mu(U_r(x))/\lambda_d(U_r(x))$, where λ_d is Lebesgue measure on $\mathbf{R}^d$, and let

$$m_u(x) = \sup\{A_r(y) : r > 0, \|x-y\| < r\}\,.$$

Show that m_u is a lower semi-continuous function on $\mathbf{R}^d$.

Show that if $\alpha > 0$ then $\lambda_d(\{x : m_u(x) > \alpha\}) \leqslant 3^d/\alpha$.

Describe briefly how this result can be used.

4

Suppose that (X, d) is a compact metric space and that $\mathbf{P}$ and $\mathbf{Q}$ are Borel probability measures on X. Show that there exists a Borel probability measure π_0 on $X \times X$, with marginals $\mathbf{P}$ and $\mathbf{Q}$, such that

$$\int_{X \times X} d(x,y)\, d\pi_0(x,y) = m_d = \sup\{\int_X f\, d\mathbf{P} + \int_X g\, d\mathbf{Q} : (f,g) \in J\}$$

where

$$J = \{(f,g) \in C(X) \times C(X) : f(x) + g(y) \leqslant d(x,y) \text{ for } (x,y) \in X \times X\}.$$

END OF PAPER