UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 31 May, 2013 9:00 am to 11:00 am

PAPER 69

FLUID DYNAMICS OF ENERGY SYSTEMS

*There are **SIX** questions in total.*

You may attempt any number of questions.

*Full marks can be gained from complete answers to **THREE** questions.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Fluid of viscosity μ passes through a permeable two-dimensional porous rock from the point $x = 0$ to $x = L$, in the vertical region $0 < y < H$.

The permeability has value $k = k_0 + k_1\, y/H$.

If a positive pressure drop ΔP is applied from $x = 0$ to $x = L$, calculate the mean travel time for fluid to migrate across the layer. You may assume the flow is all directed parallel to the x axis.

Calculate the effective macro-dispersion coefficient D_m associated with the shear in the flow, and describe whether you expect this to be significant in the migration of the fluid from $x = 0$ to $x = L$ if the pore scale dispersion coefficient is D_p.

In the case that the dispersion controls the mixing in the flow, derive an expression for the concentration with time of a tracer arriving at $x = L$ if the tracer is released into the flow at $x = 0$ at a constant flux for $t > 0$.

2

A volume of fluid V per unit length along a fracture fills an aquifer during a storm as fluid flows down a fracture and into the aquifer, of permeability k. If the fluid is then able to spread into the aquifer, as a two-dimensional flow, and also flow back into the fracture, show that the dipole moment of the flow, D, is conserved. Here D is defined as

$$D = \int_0^\infty hx dx$$

and $h(x,t)$ is the depth of the flow, with x the perpendicular horizontal distance from the fracture along the aquifer. You may assume the aquifer is initially dry, has an impermeable and horizontal lower boundary, and the flow only spreads in the region $x > 0$.

Show that the volume of fluid V decays at a rate given by

$$\frac{1}{V}\frac{dV}{dt} = \frac{1}{2t},$$

and show the solution for the shape of the current $h(x,t)$ has the form

$$\begin{aligned} h &= at^{\alpha} f(\eta)\,, \\ f &= b\eta^2 + c\eta^{1/2}\,, \\ \eta &= \frac{x}{dt^{\beta}}\,, \end{aligned}$$

where a, α, b, c, d, β are constants to be found.

3

CO_2 is injected into an aquifer with a sloping boundary, and takes up a shape

$$h = \begin{cases} ax & 0 < x < L \\ a(2L - x) & L < x < 2L\,, \end{cases}$$

where x is the distance along the sloping boundary and h is the depth of the current in the direction perpendicular to the boundary. The CO_2 migrates upslope with speed u under buoyancy, and as the tail of the current migrates, a fraction s of the CO_2 is left in the rock owing to capillary retention.

Show that the current satisfies the relations

$$\phi(1-s)\frac{\partial h}{\partial t} = -u\frac{\partial h}{\partial x} \qquad \text{if} \qquad \frac{\partial h}{\partial t} < 0\,,$$

$$\phi\frac{\partial h}{\partial t} = -u\frac{\partial h}{\partial x} \qquad \text{if} \qquad \frac{\partial h}{\partial t} > 0\,.$$

Using the method of characteristics, draw a diagram to illustrate how the current height evolves with time as it migrates upslope. Hence calculate the region of the formation below the upper boundary in which CO_2 is trapped by capillary retention.

4

Chemical of density $\rho + \Delta\rho$ is added at a volume flow rate Q_s to the top of a cylindrical reactor vessel with a radius d and a vertical axis containing fluid of density ρ.

Assuming the fluids mix as a result of the buoyancy force, use dimensional arguments to show that the mixing is driven by a velocity of the form

$$u \approx \left[\frac{dg'}{dz} d^2\right]^{1/2},$$

where dg'/dz is the vertical buoyancy gradient in the reactor with g' the reduced gravity of the dense mixed fluid relative to the original fluid in the reactor. [Note that in this expression z increasing upwards — it is convenient to denote the top of the reactor as the point where $z = 0$, so the reactor vessel lies in the region $z < 0$.]

Show that the instantaneous flux of reduced gravity in the reactor vessel is given by

$$F = d^2 \frac{d}{dz}\left[d^2 \left(\frac{dg'}{dz}\right)^{3/2}\right].$$

Show also that the equation for the rate of change of reduced gravity at each point in the reactor, in the case that there is a steady upflow Q of the original fluid in the reactor, is given by

$$d^2 \frac{\partial g'}{\partial t} + Q \frac{\partial g'}{\partial z} = d^4 \frac{\partial}{\partial z}\left[\left(\frac{\partial g'}{\partial z}\right)^{3/2}\right].$$

A steady state solution to this equation exists in which the descending fluid is arrested by the upwards flow such that $g' = 0$ below some height $z = -H$. What condition must dg'/dz satisfy at $z = -H$? Determine the steady solution for g' in $-H < z < 0$.

Calculate the distance H in terms of the volume flux of dense fluid supplied to the top of the reactor Q_s, assuming that at the top of the reactor all the dense fluid supplied from the source mixes downwards by the turbulent diffusion.

Determine the effective eddy diffusivity of a passive tracer which would be mixed downwards by this flow in the region $-H < z < 0$.

5

The floor of a naturally ventilated building has a distributed source of hot buoyant fluid with heat flux Q and negligible mass flux. Openings of area A are located on opposite sides at the top and base of the building. Wind blows over the building generating a pressure difference ΔP across the building.

Derive an equation for the temperature of the building relative to the exterior for both the case in which the wind-generated pressure difference acts in the same direction and the case when it acts in the opposite direction to the buoyancy force.

For the case in which the wind-generated pressure difference acts in opposition to the buoyancy force, there is a range of conditions for which there are three solutions to the flow problem. Discuss the stability of these solutions.

Explain how the system evolves if the wind speed increases or if the heat flux supplied to the system decreases. Do these have the same effect?

6

In a naturally ventilated space a point source of buoyancy at floor level generates a turbulent plume with buoyancy flux B and negligible mass flux. The space has height H and two openings each of effective area A, one in the floor and one in the ceiling. Show that a warm layer with buoyancy g' develops in the upper part of the space a height h above the floor where

$$A\sqrt{g'(H-h)/2} = \lambda B^{1/3} h^{5/3}$$

and

$$B = A g' \sqrt{g'(H-h)/2}\,.$$

Use these expressions to derive an equation for the height of the interface.

In some cases, the source of buoyancy also supplies a volume flux Q. Evaluate the critical value of Q for which the natural ventilation flow is blocked, so that there is no inflow through the lower opening.

[You may assume that the volume flux in a turbulent buoyant plume at a height h above a point source of buoyancy B is given by $\lambda B^{1/3} h^{5/3}$.]

END OF PAPER