UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS   Part III

Monday, 3 June, 2013   9:00 am to 12:00 pm

PAPER 68

SLOW VISCOUS FLOW

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $\mathbf{u}$ be a Stokes flow (with no body force) in a region V with bounding surface ∂V and outward normal $\mathbf{n}$. Given that the local rate of viscous dissipation is $\boldsymbol{\sigma} : \mathbf{e}$, where $\boldsymbol{\sigma}$ is the stress tensor and $\mathbf{e}$ the strain-rate tensor, show that the total dissipation is given by

$$D = \int_{\partial V} \mathbf{u} \cdot \boldsymbol{\sigma} \cdot \mathbf{n} \, dS.$$

Let $\mathbf{u}_0$, with stress tensor $\boldsymbol{\sigma}_0$, be the Stokes flow in V_0 due to a specified velocity $\mathbf{U}_0(\mathbf{x})$ on ∂V_0. Now let $\mathbf{u}_0 + \mathbf{u}'$, with stress tensor $\boldsymbol{\sigma}_0 + \boldsymbol{\sigma}'$, be the Stokes flow produced by adding a force-free, couple-free, rigid particle to the flow while maintaining the velocity equal to $\mathbf{U}_0(\mathbf{x})$ on ∂V_0. Show that the increase D' in dissipation due to the presence of the particle is given by

$$D' = \int_{\partial V_0} \mathbf{U}_0 \cdot \boldsymbol{\sigma}' \cdot \mathbf{n} \, dS \ .$$

Use the Reciprocal Theorem to show further that

$$D' = \int_A (\mathbf{u}_0 \cdot \boldsymbol{\sigma}' - \mathbf{u}' \cdot \boldsymbol{\sigma}_0) \cdot (-\mathbf{n}) \, dS,$$

where A is the surface of the particle and $-\mathbf{n}$ is its outward normal.

A force-free, couple-free, rigid sphere of radius a is placed at the origin in an unbounded strain flow with uniform strain rate $\mathbf{E}$. Find the perturbation to the flow arising from the presence of the sphere. Given that the total stress on the surface of the sphere is

$$\boldsymbol{\sigma}_0 + \boldsymbol{\sigma}' = 5\mu\big\{(\mathbf{n} \cdot \mathbf{E} \cdot \mathbf{n})(\mathbf{I} - 2\mathbf{n}\mathbf{n}) + (\mathbf{E} \cdot \mathbf{n})\mathbf{n} + \mathbf{n}(\mathbf{E} \cdot \mathbf{n})\big\},$$

calculate the increase in dissipation due to the presence of the sphere.

[*You may assume that the difference between the unbounded situation and the case* $\mathbf{U}_0 = \mathbf{E} \cdot \mathbf{x}$ *on* $r = R$, *where* $R \gg a$, *is negligible near the sphere.*]

A volume fraction ϕ of such spheres is now distributed throughout the straining flow, where $\phi \ll 1$ so that interaction between the spheres can be neglected. Calculate the number of spheres per unit volume and deduce that the average increase in dissipation per unit volume is such that the fluid appears to have an effective viscosity

$$\mu_{\text{eff}} = \mu(1 + \tfrac{5}{2}\phi).$$

2

State and prove the Reciprocal Theorem for two Stokes flows with viscosity μ and no body force.

Prove that the resistance matrix, giving the force $\mathbf{F}$ and couple $\mathbf{G}$ exerted *by* a rigid body when moving with velocity $\mathbf{U}$ and angular velocity $\boldsymbol{\Omega}$ through surrounding viscous fluid, otherwise at rest, is both symmetric and positive definite.

A rigid body comprises three point masses with weights mg at $O = (0,0,0)$, λmg at $A = (2L, 0, 0)$ and λmg at $B = (0, 2L, 0)$, joined along OA and OB by two thin rods of negligible weight, length $2L$ and thickness ϵL. The hydrodynamic resistance to motion of the point masses is negligible and that of the thin rods is given by the slender-body formula

$$\mathbf{f}(\mathbf{X}) = C(\mathbf{I} - \tfrac{1}{2}\mathbf{X}'\mathbf{X}') \cdot \dot{\mathbf{X}}$$

where $C = 4\pi\mu/|\ln\epsilon|$ and $\mathbf{X}(s,t)$ is the position along the rod. Calculate the 6×6 resistance matrix for this body with respect to the axes fixed in the body. Verify that

$$CL \begin{pmatrix} U_x \\ U_y \\ \Omega_z L \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{6} & \frac{1}{4} \\ -\frac{1}{6} & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} F_x \\ F_y \\ G_z/L \end{pmatrix} .$$

The body is allowed to fall under gravity from an initial position in which OA is horizontal and B is vertically above O. In the subsequent motion OA makes an angle $\theta(t)$ above horizontal. Show that

$$CL^2\dot{\theta} = \tfrac{1}{4}(1-\lambda)mg(\cos\theta - \sin\theta) .$$

Deduce that if $\lambda < 1$ then $\theta(t)$ increases monotonically from 0 to $\pi/4$ as $t \to \infty$. What happens if $\lambda > 1$?

Show that the horizontal velocity component U_h of the point O satisfies

$$CLU_h = \tfrac{1}{6}(1-\lambda)mg(\cos^2\theta - \sin^2\theta).$$

Deduce that if $\lambda < 1$ then the point O drifts sideways as it falls by a total horizontal distance $2L/3$ in the direction of the initial orientation of OA. Find the corresponding result for $\lambda > 1$.

Sketch the fall of the body for $\lambda < 1$ and $\lambda > 1$, showing both its trajectory and orientation.

3

A thin layer of viscous fluid of thickness $h(x,t)$ lies between a hot rigid boundary at $z=0$ and a cold inviscid environment. The temperature $T(x,z,t)$ of the fluid satisfies

$$\frac{DT}{Dt} = \kappa\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right) \quad \text{in} \quad 0 < z < h(x,t) \ , \tag{1}$$

$$T = T_0 \text{ at } z = 0 \ , \qquad -\kappa\frac{\partial T}{\partial z} = \alpha[T - (T_0 - \Delta T)] \text{ at } z = h(x,t) \ ,$$

corresponding to a fixed boundary temperature T_0 and conductive cooling to an environmental temperature $T_0 - \Delta T$. Here α and κ are constants.

Find the steady temperature distribution $T(z)$ when h is uniform and the fluid velocity $\mathbf{u}$ is zero. Show that if $\alpha h/\kappa \ll 1$ then

$$T(h) \approx T_0 - \Delta T\frac{\alpha h}{\kappa} \ . \tag{2}$$

Use scaling on the terms in (1) to show that (2) still holds when h varies on a lengthscale L and the x-component of $\mathbf{u}$ has typical magnitude U, provided that $\epsilon^2 \equiv (h/L)^2 \ll 1$ and $Pe \equiv Uh^2/(\kappa L) \ll 1$.

The fluid has a temperature-dependent surface tension $\gamma(T) = \gamma_0 - \gamma'(T - T_0)$, where $\gamma' > 0$ is a constant. The other properties of the fluid are independent of temperature, and gravity is negligible. Assuming that the surface temperature is given by (2), use lubrication theory to show that

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\gamma_0}{3\mu}h^3\frac{\partial^3 h}{\partial x^3} + \frac{\gamma'\Delta T\alpha}{2\kappa\mu}h^2\frac{\partial h}{\partial x}\right) = 0 \ . \tag{3}$$

[*Justification of the approximations in lubrication theory is* **not** *required.*] You may assume that the surface curvature is approximately $\partial^2 h/\partial x^2$, but the use of γ_0 in the second term instead of $\gamma(T)$ should be justified by a scaling argument.

Give a brief *physical* explanation, with diagrams, why the second and third terms in (3) cause perturbations to a uniform film thickness to decay and to grow.

Equation (3) can be reduced to the dimensionless form

$$H_\tau + (H^2 H_X)_X + (H^3 H_{XXX})_X = 0 \tag{4}$$

by defining $H = h/\hat{h}$, $X = \epsilon x/\hat{h}$ and $\tau = t/\hat{t}$. Find the timescale $\hat{t}$ and aspect ratio ϵ. Obtain and sketch the dispersion relationship $s(k)$ for small disturbances of the form $H = 1 + \delta\exp(s\tau + ikX)$ with $\delta \ll 1$. What is the most unstable wavenumber?

Show that steady solutions of (4) with zero net flux satisfy

$$\tfrac{1}{2}H_X^2 + V(H) = E,$$

where $V(H) = H(\ln H - 1)$ and E is a constant, provided that $\hat{h}$ has been chosen so that $H = 1$ when $H_{XX} = 0$. By sketching $V(H)$, or otherwise, describe the form $H(X)$ of the steady solutions for $E = -1$, $-1 < E < 0$ and $E = 0$. How do these solutions relate to the dispersion relationship found earlier as $E \to -1$?

4

Fluid of viscosity μ and density $\rho + \Delta\rho$ spreads as a gravity current beneath fluid of density ρ and over a rigid horizontal surface $z = 0$. A constant uniform shear stress τ is exerted on the upper surface of the gravity current by some mechanism (such as an imposed background flow in the upper fluid). Surface tension is negligible.

Assuming that the gravity current can be described by lubrication theory, show that its thickness $h(x, y, t)$ obeys

$$\frac{\partial h}{\partial t} + \frac{\tau}{2\mu}\frac{\partial h^2}{\partial x} = \frac{\Delta\rho\, g}{3\mu}\boldsymbol{\nabla}\cdot\left(h^3\boldsymbol{\nabla}h\right),$$

where x and y are the horizontal coordinates parallel and perpendicular to the shear stress τ. If h has typical magnitude H and varies on a horizontal lengthscale L, what dimensionless groups, expressed in terms of the given parameters, must be small for the approximations of lubrication theory to hold?

Consider the case of steady flow from a point source of constant volume flux Q located at the origin. At large distances x downstream, the thickness $h(x, y)$ and cross-stream width $2y_N(x)$ of the current satisfy $h \ll y_N \ll x$. By making suitable approximations and scaling estimates, find how y_N depends on x and the other parameters. Hence find a similarity solution for $h(x, y)$ and determine the corresponding $y_N(x)$.

Sufficiently close to the source, the similarity solution does not apply since the lengthscales in the x and y directions are comparable. Use scaling arguments to estimate the distance x_N that the current spreads upstream of the source, explaining the balance that determines this distance.

Now consider the case of steady flow with no variation in the y-direction from a line source at $x = 0$ with constant volume flux Q_{2d} per unit width. Assuming that h is a constant in $x > 0$, determine $h(x)$ in $x < 0$ and deduce the value of x_N.

END OF PAPER