UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 4 June, 2013 9:00 am to 12:00 pm

PAPER 67

PERTURBATION AND STABILITY METHODS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Find an asymptotic expansion for the roots x_n of the equation

$$\tan x = \mu x$$

with both x and μ real, and $x_n \to \infty$. Your expansion should include the first term that depends on μ.

What restriction, if any, must be placed on μ for validity of the expansion?

(b) Explain briefly the class of (real) integrals appropriate for use of (i) Watson's lemma and (ii) Laplace's method. Use Laplace's method to find **two** terms of an asymptotic expansion for the Gamma Function

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \mathrm{d}t$$

in the limit $x \to \infty$.

[If

$$I_n = \int_{-\infty}^{\infty} s^{2n} e^{-s^2/2} \mathrm{d}s$$

then $I_0 = \sqrt{2\pi}$ and $I_n = (2n-1)I_{n-1},\ n = 1, 2, ...$]

2

Explain briefly what is meant by the **method of multiple scales**.

(a) Two oscillators $x(t), y(t)$ are weakly coupled so that for small ϵ,

$$\begin{aligned} \ddot{x} + x + \epsilon(\alpha x + xy) &= 0 \,, \\ \ddot{y} + n^2 y + \epsilon xy &= 0 \,, \end{aligned}$$

and $x(0) = X$, $\dot{x}(0) = 0$, $y(0) = Y$, $\dot{y}(0) = 0$.

Suppose that $n = 2$. Use the method of multiple scales to find leading order solutions for $x(t)$ and $y(t)$, and specify the range of t for which they are valid. Under what circumstances do oscillations grow?

Suppose instead that $n = 1$. On what timescale could oscillations of the system grow? [Full solutions for $x(t)$ and $y(t)$ are **not** required in this case.]

(b) Derive the leading order solution of the WKB equation

$$y'' + k^2 y = 0 \,,$$

where k depends on ϵx and $\epsilon \to 0$. Specify restrictions on its validity.

3

Consider the equation

$$\epsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (1+x)^2 \frac{\mathrm{d}y}{\mathrm{d}x} + y = 0 \quad \text{with} \quad y(0) = y(1) = 1\ .$$

Find the first two nonzero terms in the inner and outer expansions in the limit $\epsilon \to 0$. Determine a uniformly-valid approximation to y which is correct up to and including $O(\epsilon)$.

Describe **briefly** how you would determine the asymptotic solution of :

(i)

$$\epsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - (1+x)^2 \frac{\mathrm{d}y}{\mathrm{d}x} + y = 0 \quad \text{with} \quad y(0) = y(1) = 1\ ;$$

(ii)

$$\epsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (1+x)^2 \frac{\mathrm{d}y}{\mathrm{d}x} + y = 0 \quad \text{with} \quad y(-1) = y(1) = 1\ ;$$

(iii)

$$\epsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x \frac{\mathrm{d}y}{\mathrm{d}x} + y = 0 \quad \text{with} \quad y(-1) = y(1) = 1\ .$$

In each case you should state the location and size of each asymptotic region, but further detailed calculations of the expansions are **not** required.

4

(a) Consider the fourth-order equation

$$\frac{\partial \eta}{\partial t} + \alpha \frac{\partial^2 \eta}{\partial x^2} + \beta \frac{\partial^4 \eta}{\partial x^4} + \gamma \eta = 0 \ ,$$

where α, β, γ are nonzero real constants. By considering solutions which are proportional to $\exp(-\mathrm{i}\omega t + \mathrm{i}kx)$, derive a condition on β for the system to possess a finite maximum temporal growth rate over all real k, and explain why this is important for application of the Briggs-Bers technique. Determine necessary conditions for the occurrence of absolute instability. Why are these conditions not sufficient?

(b) In the following, the Fourier transform is defined to be

$$\tilde{f}(k) = \int_{\infty}^{\infty} \exp(-\mathrm{i}kx) f(x) \mathrm{d}x \ ,$$

and you are given that the Fourier transform of $\mathrm{H}(x)x^{\alpha}$ is

$$\frac{\exp[-\mathrm{i}\pi(\alpha+1)\mathrm{sgn}(k)/2]\alpha!}{|k|^{\alpha+1}} \quad \text{for} \quad \alpha > -1 \ .$$

Find the first two non-zero terms (if they exist) in the large-k expansion of the Fourier transforms of the following functions:

(i)

$$\frac{1}{1+|x|^3} \ ;$$

(ii)

$$\frac{1}{1+x^3} \ .$$

END OF PAPER