UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 5 June, 2013 1:30 pm to 4:30 pm

PAPER 66

BIOLOGICAL PHYSICS

*Attempt **ALL** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

An elastic filament with bending modulus A and length L has small-amplitude excursions $h(x)$ from the x-axis, and is characterized by the bending energy

$$\mathcal{E} = \frac{1}{2}\int_0^L dx A h_{xx}^2 \ .$$

a) Show that if the boundary conditions on the filament ends are taken to be identical, then there are four distinct conditions that render the Euler-Lagrange operator self-adjoint. Explain how the terminology *free-free, clamped-clamped, hinged-hinged*, and *torqued-torqued* applies to these cases.

b) From general principles we know that the set of eigenfunctions of such an operator define a complete set of basis functions. Show that these can be written as

$$W^{(n)}(x) = A\cos(k^{(n)}x) + B\sin(k^{(n)}x) + D\cosh(k^{(n)}x) + E\sinh(k^{(n)}x) \ ,$$

and find the transcendental equation satisfied by $k^{(n)}$ for the case of *clamped-clamped* boundary conditions. By a graphical construction or otherwise give approximate values for the infinite sequence of wave numbers $k^{(n)}$.

c) Use the principle of equipartition to find the variance of $h(x)$, using the expansion $h(x) = \sum a_n W^{(n)}(x)$.

d) Suppose the filament is now subject to a spatially-varying tension $\sigma(x)$, with $\sigma(0) = \sigma(L) = 0$, so that the energy functional is now

$$\mathcal{E} = \frac{1}{2}\int_0^L dx \left\{ A h_{xx}^2 + \sigma(x) h_x^2 \right\} \ .$$

Find the Euler-Lagrange equation for this functional, and show how the modal decomposition necessary to apply equipartition can still be carried through formally (i.e. without solving explicitly for the modes).

2

The original Keller–Segel model for pattern formation in the *Dictyostelium* system in two spatial dimensions involves the amoebae concentration a and the chemoattractant concentration ρ in the coupled system

$$\begin{aligned}\frac{\partial a}{\partial t} &= D_2\nabla^2 a - \nabla\cdot(D_1\nabla\rho)\ ,\\ \frac{\partial \rho}{\partial t} &= D_\rho\nabla^2\rho - k(\rho)\rho + af(\rho)\ .\end{aligned}$$

Here, $k(\rho)$ is the degradation rate of the chemoattractant, $f(\rho)$ is its production rate per amoeba, and $D_\rho, D_1(a,\rho), D_2(a,\rho)$ are all positive. Assume there exists a fixed point of the homogeneous system with values (a_0,ρ_0). Perform a linear stability analysis around this point for two-dimensional perturbations of the form $\exp(i\mathbf{q}\cdot\mathbf{x})$ and show that the condition for instability can be expressed as

$$\frac{D_1 f(\rho_0)}{D_2\kappa} + \frac{a_0 f'(\rho_0)}{\kappa} > 1\ , \tag{1}$$

where $\kappa = k(\rho_0) + \rho_0 k'(\rho_0)$, $f'(\rho) = df(\rho)/d\rho$ and $k'(\rho) = dk(\rho)/d\rho$. What is the wavelength of the fastest-growing mode?

Explain physically the competing effects embodied in each of the two ratios in the stability criterion. Which competition leads to aggregation?

3

The radially symmetric spread of an insect population density $n(r,t)$ in the plane is described by the equation

$$\frac{\partial n}{\partial t} = \frac{D_0}{r}\frac{\partial}{\partial r}\left[r\left(\frac{n}{n_0}\right)^2\frac{\partial n}{\partial r}\right]\ , \tag{$*$}$$

where D_0 is a constant diffusivity and n_0 is a constant. Suppose Q insects are released at $r=0$ at $t=0$. We wish to find a similarity solution to $(*)$ in the form

$$n(r,t) = \frac{n_0}{\lambda^2(t)}F(\xi)\ ;\quad \xi = \frac{r}{r_0\lambda(t)}\ .$$

Find the condition on $\lambda(t)$ that reduces the PDE (*) to an ODE, and then obtain $\lambda(t)$ and F such that $F(0)=1$ and $F(\xi)=0$ for $\xi \geqslant 1$. Determine r_0 in terms of Q. Sketch the function $n(r,t)$ at various times to indicate its qualitative behaviour.

END OF PAPER