UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 30 May, 2013 1:30 pm to 4:30 pm

PAPER 65

FLUID DYNAMICS OF THE ENVIRONMENT

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

*This is an **OPEN BOOK** examination.*
Candidates may only bring into the examination handwritten or personally typed lecture notes and handouts from this course.
No other material, or copies thereof, are allowed.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Consider a two-dimensional inviscid linear surface gravity wave in an infinitely deep layer of fluid $z \leqslant 0$. You may assume that the velocity potential is

$$\phi = a\frac{\omega}{k}e^{kz}\sin(kx - \omega t); \quad \omega^2 = gk,$$

and that the Lagrangian position of a fluid parcel is governed by the system of equations

$$\frac{dX}{dt} = u(X, Z, t), \quad X(0) = x; \quad \frac{dZ}{dt} = w(X, Z, t), \quad Z(0) = z.$$

(i) Show that to leading order for small time t,

$$X - x \simeq -ae^{kz}\sin(kx - \omega t), Z - z \simeq ae^{kz}\cos(kx - \omega t).$$

(ii) By considering a Taylor-series expansion or otherwise, show that there is a non-zero Stokes drift:

$$u_s = u(X, Z, t) - u(x, z, t) = a^2 k\omega e^{2kz}.$$

(b) Now consider a Boussinesq, stratified, stationary, inviscid fluid with constant buoyancy frequency N. You may assume that there is a linear monchromatic internal wave with perturbation vertical velocity:

$$w' = w_0 \exp[i(kx + ly + mz - \omega t)] = w_0 \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$$

where w_0, $\mathbf{k}$ and ω are all real.

(i) From the equations of motion or otherwise, derive explicit expressions for the two horizontal components u', v' of the perturbation velocity, the perturbation pressure p' and the perturbation density ρ' associated with this wave.

(ii) Demonstrate that the group velocity $\mathbf{c}_g$ and the phase velocity $\mathbf{c}_p$ are orthogonal.

(c) Now consider a stratified stationary fluid with constant buoyancy frequency N and non-zero kinematic viscosity ν. Let ξ be the along-beam coordinate in the direction of the group velocity. You may assume that the along-beam velocity in a monochromatic wave beam can be written as

$$U(\xi, t) = U_0 \exp\left[\frac{-\nu k^3 \xi}{2N\sin\theta}\right]\cos\omega t,$$

where θ is the angle the wavenumber vector makes with the horizontal. Calculate the leading order along-beam Stokes drift for such a wave beam and demonstrate that it vanishes when integrated over a wave period.

2

Consider a two-dimensional inviscid flow with a piecewise linear mean velocity distribution $\overline{U}(z)$ and and piecewise constant mean density distribution $\overline{\rho}(z)$. You may assume that the perturbation streamfunction

$$\psi(x,z,t) = Re\left[\hat{\psi}(z)\exp(ik[x-ct])\right]$$

(the wavenumber k is real) satisfies $\hat{\psi}'' = k^2\hat{\psi}$ except at "interfaces" where the mean velocity, density or vorticity is discontinuous. At such interfaces, you may assume that $\hat{\psi}$ satisfies the jump conditions:

$$\left[\frac{\hat{\psi}}{(\overline{U}-c)}\right]_-^+ = 0; \quad \left[(\overline{U}-c)\frac{d}{dz}\hat{\psi} - \hat{\psi}\frac{d}{dz}\overline{U} - \frac{g\overline{\rho}}{\rho_0}\left(\frac{\hat{\psi}}{(\overline{U}-c)}\right)\right]_-^+ = 0.$$

(a) Using a characteristic length scale h, velocity difference ΔU, density difference $\Delta\rho$ and reference density ρ_0 to scale quantities

$$c = \frac{\Delta U}{2}\tilde{c},\ \overline{U} = \frac{\Delta U}{2}\tilde{U},\ \overline{\rho} = \rho_0 + \frac{\Delta\rho}{2}\tilde{\rho},\ z = \frac{h\tilde{z}}{2},\ \alpha = \frac{kh}{2},\ J = \frac{g\Delta\rho h}{\rho_0\Delta U^2},$$

show that the nondimensional forms of the jump conditions are

$$\left[\frac{\tilde{\psi}}{(\tilde{U}-\tilde{c})}\right]_-^+ = 0; \quad \left[(\tilde{U}-\tilde{c})\frac{d}{d\tilde{z}}\tilde{\psi} - \tilde{\psi}\frac{d}{d\tilde{z}}\tilde{U} - J\tilde{\rho}\left(\frac{\tilde{\psi}}{(\tilde{U}-\tilde{c})}\right)\right]_-^+ = 0.$$

(b) Now consider a three-layer flow with constant linear velocity shear, such that

$$\overline{U} = \frac{\Delta U z}{h}, \qquad \rho = \begin{cases} \rho_0 + \frac{\Delta\rho}{2} & z < \frac{-h}{2}; \\ \rho_0 & |z| < \frac{h}{2}; \\ \rho_0 - \frac{\Delta\rho}{2} & z > \frac{h}{2}. \end{cases}$$

(i) Show that $\tilde{c}$ satisfies

$$\tilde{c}^4 - \left(2 + \frac{J}{\alpha}\right)\tilde{c}^2 + \frac{(2\alpha - J)^2 - J^2 e^{-4\alpha}}{4\alpha^2} = 0.$$

(ii) Hence show that the flow is unstable for

$$\frac{2\alpha}{1+e^{-2\alpha}} < J < \frac{2\alpha}{1-e^{-2\alpha}}.$$

(iii) Interpret this instability in terms of a wave resonance in the limit of large wavenumber.

3

A lake forms on the top of the Greenland ice sheet during the summer melt season with meltwater at temperature $T_s > T_m$, where T_m is the melting temperature. When the lake contains a volume V of water it fractures the glacial ice and instantaneously drains to the base where it spreads along the ice-rock interface as a turbulent, two-dimensional gravity current.

(a) Construct a model of the sub-glacial gravity current on a horizontal bedrock. The pressure at the ice-water interface, $p = p_0$, is constant, and flow is driven principally by gradients in the hydrostatic pressure within the current. Develop a model (including diagram) of the characteristic turbulent velocity u by balancing the depth-averaged pressure gradient with turbulent shear stresses along the walls of the form

$$2\tau = 2\mu \frac{u}{\delta_v} \simeq \int_0^h -\frac{\partial p}{\partial x} dz,$$

where μ is the dynamic viscosity of the fluid, and δ_v is the **constant** characteristic depth of the viscous boundary layer.

Show that the resultant shallow-water model of the height of the current $h(x,t)$ is of the form

$$\frac{\partial h}{\partial t} + \gamma \frac{\partial}{\partial x}\left(h^2 \frac{\partial h}{\partial x}\right) = 0,$$

where γ is a constant expressed in term of physical parameters of the system.

(b) Find the self-similar height and extent of the meltwater pulse as a function of time.

(c) Now consider the impact of melting on the evolution of the turbulent current. Use an energy balance argument across the melting/freezing interface to derive an expression for the melt rate v_m, assuming that the width of the thermal boundary layer $\delta_T \sim \delta_v (\sim \text{constant})$, and that the temperatures of the meltwater and glacier are constants T_s and T_m respectively. The constant, uniform temperatures of the meltwater and glacier are T_s and T_m, respectively.

(d) Write down the equation governing the evolution of the profile of the gravity current, and show how the rate of change in the volume of the meltwater cavity is related to the melt rate and rate of advance of the current.

4

Convection, initiated from a hot, localised source in an otherwise infinite porous medium saturated with fluid at temperature T_0 develops into a plume fed by constant heat flux F_h, in which the key balance is between advection and diffusion of heat. After an initial transient, a steady-state plume develops whose vertical extent is very much greater than its width. The density of the fluid is given by the linear equation of state

$$\rho = \rho_0[1 - \alpha(T - T_0)],$$

and therefore the vertical rise of fluid is given by a natural buoyancy velocity

$$w = -\frac{g\Delta\rho\Pi}{\mu},$$

where $\Delta\rho = \rho(x, z) - \rho_0$, Π is the permeability of the porous medium, and μ the dynamic viscosity of the fluid.

Show that the shape and extent of the steady-state plume can be modelled by the steady-state advection, and predominantly lateral diffusion, of heat

$$u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial z} \simeq \kappa\frac{\partial^2 T}{\partial x^2},$$

where κ is the constant thermal diffusivity. You may find it useful to introduce the streamfunction

$$(u, w) = (\psi_z, -\psi_x).$$

Additionally, show that the convective heat flux, F_h, is constant with height and equal to the total rate of heat input at the source. Determine the self-similar scaling for the width, velocity and temperature within the plume, and determine the self-similar profile. Using your steady-state profile, estimate the rise velocity of the head of the plume.

END OF PAPER