MATHEMATICAL TRIPOS Part III

Monday, 10 June, 2013 1:30 pm to 3:30 pm

PAPER 64

IMAGE PROCESSING — VARIATIONAL AND PDE METHODS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $\Omega = (a,b)^2$ a square image domain and $u \in L^1_{loc}(\Omega)$ an image function. State the definition of the total variation of u and define the space $BV(\Omega)$ with corresponding norm.

Prove that $BV(\Omega)$ is a Banach space. [*Hint: You may assume here that* $BV(\Omega)$ *is a normed space and that the total variation is lower-semicontinuous with respect to the* L^1*-norm.*]

Now, let $\Omega = \mathbb{R}^2$ and $g(x)$, $x \in \Omega$ be the characteristic function of a ball with centre in the origin and radius R, $0 < R < \infty$. Derive an explicit formula for the ROF-minimiser u, that is for

$$u = \operatorname{argmin}_v \left\{ \alpha |Dv|(\Omega) + \frac{1}{2} \|v - g\|_2^2 \right\}.$$

Carefully justify each step of your derivation and state (without proof) all the theorems you use.

2

Let $\Omega = (0,1)^2$ a square image domain, $1 > g > 0$ be a bounded image function, and T a linear, continuous and positivity preserving operator from $L^1(\Omega)$ to $L^1(\Omega)$. Moreover assume that $T\chi_\Omega \neq 0$.

Prove that for $u \in \{v \in L^1(\Omega),\ \log v \in L^1(\Omega)\}$ the function $\phi(u,g) = Tu - g\log(Tu+1)$ is convex, and that it fulfils the following coercivity condition

$$\int_\Omega (Tu - g\log(Tu+1))\ dx \geqslant \|Tu\|_1 - \|g\|_\infty \cdot \log\|Tu+1\|_1 .$$

[*Hint: You may use the fact that for a concave function* $f : \mathbb{R} \to \mathbb{R}$ *on the real line we have that* $f\left(\int_\Omega u(x)\ dx\right) \geqslant \int_\Omega f(u(x))\ dx$*.*]

Now, consider the following variational problem

$$\min_{u\in L^1(\Omega),\log(u)\in L^1(\Omega)} \left\{\alpha|Du|(\Omega) + \int_\Omega (Tu - g\log(Tu+1))\ dx\right\}$$

and prove existence of solutions for the above problem. When is a solution unique? Justify all your steps.

[*Hint: You may use, without proof, Rellich's compactness theorem and the following form of the Poincaré-Wirtinger inequality: For* $u \in BV(\Omega)$*, let*

$$u_\emptyset := \frac{1}{|\Omega|}\int_\Omega u(x)\ dx .$$

Then there exists a constant $C > 0$ *such that*

$$\|u - u_\emptyset\|_1 \leqslant C|Du|(\Omega) .\quad]$$

In the finite dimensional setting, that is $\Omega = \{x_1, \ldots, x_M\}^2$, state (without proof) what kind of noise distribution the data fidelity $\sum_{i,j} \phi(u(x_i,x_j), g(x_i,x_j))$ models approximately?

3

For $g \in C(\mathbb{R}^2)$, bounded, consider the linear diffusion equation

$$\begin{aligned} u_t &= \Delta u \\ u(x, t = 0) &= g(x). \end{aligned}$$

Give an explicit formula for the solution $u(x,t)$ of this equation within the class of functions that satisfy

$$|u(x,t)| \leqslant M \; e^{a|x|^2}, \quad M, a > 0.$$

Relate such a solution at a time $T > 0$ with linear filtering of g with a Gaussian kernel of standard deviation σ. Investigate the effect of Gaussian filtering in the frequency domain.

Now, consider the Perona-Malik equation

$$\begin{aligned} u_t &= \mathrm{div}(c(|\nabla u|) \; \nabla u) \\ u(x, t = 0) &= g(x), \end{aligned}$$

where $c(y) = y \cdot e^{-\frac{y^2}{2\lambda^2}}$ for a positive λ. Explain the dynamics of this equation in one space dimension in terms of forward and backward diffusion in dependence of λ.

For $u \in C^\infty(\mathbb{R}^2)$ and $h > 0$ let

$$\mathrm{mean}_h(u)(x) := \frac{u(x + (h,0)) + u(x - (h,0)) + u(x + (0,h)) + u(x - (0,h))}{4},$$

be the mean of u in a (vertical-horizontal) h-neighbourhood of x. Prove, that in a point x_0 where $u(x_0) = \mathrm{mean}_h(u)(x_0)$ for all $\infty > h' > h > 0$ (with h' fixed) we have that in the limit as $h \to 0$ the function u fulfils $\Delta u(x_0) = 0$. Assuming that $Du(x_0) \neq 0$, state (without proof) the differential equation for u in x_0 that one receives when replacing the mean by the median, that is

$$\mathrm{median}_h(u)(x) = \text{median value of the set } \{u(y), \; y \in B_x(h)\},$$

where $B_x(h)$ is a disc with radius h and centre x. Based on this differential equation what can you say about the local shape of the level line of u that passes through x_0?

4

Write an essay on image segmentation using the Mumford–Shah segmentation model.

END OF PAPER