UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 31 May, 2013 1:30 pm to 4:30 pm

PAPER 63

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

Attempt no more than ***THREE*** *questions from Section I and* ***ONE*** *from Section II.*

There are ***SEVEN*** *questions in total.*

The questions in Section II carry twice the weight of those in Section I. Questions within each Section carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1

The diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad t \geqslant 0, \quad x \in [0,1],$$

given with zero Dirichlet boundary conditions and an initial condition at $t = 0$, is discretised by the finite-difference method

$$-\alpha u_{m-1}^{n+1} + (1+2\alpha)u_m^{n+1} - \alpha u_{m+1}^{n+1} = u_m^n, \qquad n \geqslant 0, \quad m = 1, \ldots, M,$$

where $\Delta x = 1/(M+1)$ and the coefficient α is allowed to depend on the Courant number.

1. Determine α so that the method is of the highest possible order.
2. Carefully justifying your steps, find the range of Courant numbers for which the highest-order method is stable.

2

Consider the two-stage implicit Runge–Kutta method with the Butcher tableau

$$\begin{array}{c|cc} \frac{1}{2} - \frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \frac{1}{2} + \frac{\sqrt{3}}{6} & \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}.$$

1. Determine the order of this method.
2. Is the method A-stable?
3. Is it algebraically stable?

3

Let H be a Hilbert space, equipped with the inner product $\langle \cdot, \cdot \rangle$ and let $\mathcal{L}$ be a bounded linear operator in H.

1. Define what is meant by $\mathcal{L}$ being self adjoint, elliptic and positive definite.
2. Suppose that $\mathcal{L}$ is positive definite. Prove that $\mathcal{L}u = f$ is the Euler–Lagrange equation of the variational problem $I(v) = \langle \mathcal{L}v, v\rangle - 2\langle f, v\rangle$ and that the weak solution of $\mathcal{L}u = f$ is the unique minimum of I.
3. Let $p(x) > 0$, $q(x) \geqslant 0$ for $x \in [-1, 1]$ and set $\mathcal{L}u = -(pu')' + qu$, acting in an appropriate Hilbert space (which you should describe) with zero Dirichlet boundary conditions. Prove that $\mathcal{L}$ is positive definite.

4

The ODE system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$, $\mathbf{y}(0) = \mathbf{y}_0$, is solved by the two-step method

$$\mathbf{y}_{n+2} - (1+a)\mathbf{y}_{n+1} + a\mathbf{y}_n = h(1-a)\mathbf{f}(\mathbf{y}_n) + \frac{h^2}{2}(3-a)\frac{\partial \mathbf{f}(\mathbf{y}_{n+2})}{\partial \mathbf{y}}\mathbf{f}(\mathbf{y}_{n+2}),$$

where a is a real constant.

1. Assuming without proof the validity of the Dahlquist Equivalence Theorem, determine the order and convergence for different values of a.
2. Letting $a = 0$, check whether the method is A-stable.

5

The linear ODE

$$\mathbf{y}' = (A+B)\mathbf{y}, \quad t \geqslant 0, \qquad \mathbf{y}(0) = \mathbf{y}_0,$$

is approximated by the Strang splitting

$$\mathbf{y}_{n+1} = e^{\frac{1}{2}hA} e^{hB} e^{\frac{1}{2}hA} \mathbf{y}_n, \qquad n = 0, 1, \ldots .$$

Prove that this is a second-order method.

Consider the convection-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\partial u}{\partial x}, \qquad t \geqslant 0, \quad x \in [-1, 1],$$

given with an initial condition at $t = 0$, $x \in [-1, 1]$ and zero Dirichlet boundary conditions. Both the space derivatives are discretised by second-order central differences and this results in an ODE system of the form $\mathbf{u}' = (A+B)\mathbf{u}$. What are the matrices A and B?

The semidiscretized system is solved with the Strang splitting. Is the outcome a stable method?

SECTION II

6

Describe the Engquist–Osher method for a single, one-dimensional, hyperbolic nonlinear conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0.$$

Prove its stability, subject to suitable conditions.

7

Write an essay on stability analysis of partial differential equations of evolution using Fourier analysis, inclusive of the influence of boundary conditions.

END OF PAPER