UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 11 June, 2013 1:30 pm to 3:30 pm

PAPER 62

CONVEX OPTIMISATION WITH APPLICATIONS IN IMAGE PROCESSING

Attempt no more than ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

a) Prove that every closed convex set $C \subseteq \mathbb{R}^n$ is the intersection of all closed half-spaces that contain C.

b) State the central theorem about the Legendre-Fenchel transform and explain the most important steps in the proof leading to the final relation between f^{**} and cl con$f$. How is the theorem in a) used in the proof?

c) Find the subdifferential of $f(x) = \max\{a^\top x, b^\top x\}$ for given $a, b \in \mathbb{R}^n$. Carefully explain each step. [*Hint: It may be helpful to rewrite the maximum as a support function.*]

2

a) Assume $f : \mathbb{R}^n \times \mathbb{R}^m \to \bar{\mathbb{R}}$ is a given perturbation function. Define the primal and dual objectives φ, ψ, primal and dual problems, and primal and dual marginal functions p, q. State a sufficient condition for strong duality.

b) The *inf-convolution* $F : \mathbb{R}^n \to \bar{\mathbb{R}}$ of two functions $h, k : \mathbb{R}^n \to \bar{\mathbb{R}}$ is pointwise defined as

$$F(z) \quad := \quad \inf_{x \in \mathbb{R}^n} \{k(x) + h(z - x)\}.$$

Assume that h, k are non-negative real-valued, i.e., $h, k : \mathbb{R}^n \to [0, +\infty)$, proper, lower semi-continuous, and convex.

Show that for every $z \in \mathbb{R}^n$ we can find a perturbation function $f : \mathbb{R}^n \times \mathbb{R}^n \to \bar{\mathbb{R}}$ with marginal function $p(u) = F(z + u)$. Prove that strong duality holds for every $z \in \mathbb{R}^n$.

c) Compute the dual objective ψ and show that the inf-convolution can be expressed through the conjugates as $F = (k^* + h^*)^*$. How can we practically compute elements of the subdifferential $\partial F(z)$, assuming k^* and h^* are known?

3

a) State Farkas' Lemma and prove it by defining suitable cones and using the Legendre-Fenchel transform. Explain any theorems that are used.

b) Consider the following system of inequalities for $x, y \in \mathbb{R}$:

$$x \geqslant 0, \quad y - x \geqslant -1, \quad x + y \leqslant -2.$$

Transform the system into a suitable standard form and use Farkas' Lemma to prove that it does not have a solution.

4

a) State the definition of forward- and backward-steps $F_{\tau f}$ and $B_{\tau f}$ in terms of set-valued mappings and motivate their definition. Show that the backward step $B_{\tau f}$ is at most single-valued for every proper, lower semi-continuous, convex function f and $\tau > 0$. Carefully explain each step of the proof. In particular, point out where the assumptions on f are used.

b) Assume that $\tau > 0$ and f is proper, lower semi-continuous, and convex. Show that a backward step $B_{\tau(f^*)}$ on the *conjugate* f^* can be computed using only a single backward step on f. Carefully explain each step.

5

a) Consider the conic problem in standard form,

$$\inf_{x \in \mathbb{R}^n} c^\top x \quad \text{s.t.} \quad Ax - b \geqslant_K 0, \tag{1}$$

where K is a proper, closed, convex, self-dual cone with associated canonical barrier function F and $A \in \mathbb{R}^{m \times n}$ has full column rank. State the dual problem. Define the primal-dual central path and state a joint characterization of the points on the primal-dual central path. Starting from this characterization, derive the Newton step for tracing the central path.

b) Consider the constrained TV-L^1 problem

$$\min_{u \in \mathbb{R}^n} \sum_{i=1}^{n} |u_i - f_i| + \lambda \sum_{i=1}^{n} \sqrt{\left(u^\top g_i^{(1)}\right)^2 + \left(u^\top g_i^{(2)}\right)^2} \quad \text{s.t.} \quad u \in [0,1]^n,$$

where $f \in \mathbb{R}^n$ discretizes the given image using n points, $\lambda \in (0, \infty)$ is a weighting parameter, and the matrices $G^{(j)} = (g_1^{(j)}, \ldots, g_n^{(j)})^\top, j = 1, 2$ discretize the directional derivatives of u in x- and y-direction.

Reformulate the problem in standard conic form (1), introducing additional variables as necessary. You do not need to define A, b and c explicitly, it is sufficient to list all constraints in a way that makes the linearity obvious.

To which class of conic problems does the reformulation belong? Find a suitable canonical barrier function for this problem.

END OF PAPER