UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 30 May, 2013 9:00 am to 12:00 pm

PAPER 61

APPROXIMATION THEORY

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

For a 2π-periodic function $f \in C(\mathbb{T})$, let $s_n(f)$ be its partial Fourier sum of degree n, and let $\sigma_n(f) = \frac{1}{n}\sum_{i=0}^{n-1} s_i(f)$ be its Fejer sum of degree $n-1$.

1) From the integral representation

$$s_n(f,x) = \frac{1}{\pi}\int_{\mathbb{T}} D_n(x-t)f(t)\,dt, \qquad D_n(x) = \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{1}{2}x},$$

derive the following expression for the Fejer kernel F_n

$$\sigma_n(f,x) = \frac{1}{\pi}\int_{\mathbb{T}} F_n(x-t)f(t)\,dt\,, \qquad F_n(x) = \frac{1}{2n}\frac{\sin^2\frac{n}{2}x}{\sin^2\frac{1}{2}x}\,.$$

Hence prove that $\|\sigma_n(f)\|_\infty \leqslant \|f\|_\infty$, carefully justifying each step.

2) Consider the so-called de la Vallee Poussin sum

$$v_{n,m}(f) := \frac{1}{m}\left(s_n(f) + s_{n+1}(f) + \cdots + s_{n+m-1}(f)\right).$$

(a) Show that, for any trigonometric polynomial t_n of degree n, and for any m, we have

$$v_{n,m}(t_n) = t_n\,.$$

(b) Find an expression for $v_{n,m}$ in terms of two Fejer sums σ_k and σ_ℓ, and use it to derive the bound

$$\|v_{n,m}(f)\|_\infty \leqslant \left(\tfrac{2n}{m}+1\right)\|f\|_\infty \quad \forall f \in C(\mathbb{T}).$$

(c) Let $\frac{n}{m} \leqslant M$. Combine (a) and (b) to establish the inequality

$$\|f - v_{n,m}(f)\|_\infty \leqslant 2(M+1)\,E_n(f) \quad \forall f \in C(\mathbb{T})\,,$$

where $E_n(f)$ is the best uniform approximation to f from $\mathcal{T}_n$, the space of all trigonometric polynomials of degree n.

2

Given $\Delta = (t_i)_{i=1}^{n+k}$, let ω_i and ψ_i be the polynomials in $\mathcal{P}_{k-1}$ defined by

$$\omega_i(x) := (x - t_{i+1}) \cdots (x - t_{i+k-1}), \qquad \psi_i(x) := \frac{1}{(k-1)!}\omega_i(x)\,,$$

and let $(N_i)_{i=1}^n$ be the corresponding B-spline sequence. From the Marsden identity

$$(x-t)^{k-1} = \sum_{i=1}^n \omega_i(x) N_i(t), \quad t_k \leqslant t \leqslant t_{n+1}, \quad \forall x \in \mathbb{R}\,,$$

show that any algebraic polynomial $p \in \mathcal{P}_{k-1}$ has the B-spline expansion

$$p(t) = \sum_{i=1}^n \lambda_i(p) N_i(t), \quad t \in [t_k, t_{n+1}]\,, \qquad (*)$$

and express the functional $\lambda_i(p)$ in terms of p, ψ_i and $x \in \mathbb{R}$. Explain briefly why the $\{\lambda_i(p)\}_{i=1}^n$ are independent of x.

Use expansion $(*)$ to prove that, for linear polynomials, we have

$$p(t) = \sum_{i=1}^n p(t_i^*) N_i(t), \qquad \forall p \in \mathcal{P}_1\,,$$

where $t_i^* = \frac{1}{k-1}(t_{i+1} + \cdots + t_{i+k-1})$.

3

For a knot sequence $(t_i)_{i=1}^{n+k} \subset [a,b]$ with distinct knots, let

$$M_i(t) := k[t_i, \ldots, t_{i+k}](\cdot - t)_+^{k-1} \quad \text{and} \quad N_i(t) := (t_{i+k} - t_i)[t_i, \ldots, t_{i+k}](\cdot - t)_+^{k-1}$$

be the sequences of L_1 and L_∞-normalized B-splines, respectively.

a) Using properties of divided differences, prove that M_i is a piecewise-polynomial function of degree $k-1$ and global smoothness C^{k-2}, with knots $(t_i, \ldots, t_{i+k})$ and with finite support $[t_i, t_{i+k}]$.

b) Using the Leibnitz rule for divided differences, or otherwise, derive the recurrence formula for B-splines:

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1} + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}\,,$$

where $N_{i,m}$ is the L_∞-normalized B-spline of order m with support $[t_i, t_{i+m}]$.

4

On $C(I)$, the space of continuous functions on $I = [-1, 1]$, the "tilde" operator $T : C(I) \to C(\mathbb{T})$ is given by the rule $\widetilde{f}(\theta) := f(\cos\theta)$, $\theta \in [0, 2\pi)$.

a) Prove that

$$\omega(\widetilde{f}, t) \leqslant \omega(f, t),$$

where $\omega(f, t)$ is the first modulus of continuity.

b) State the first Jackson theorem for periodic functions and deduce (justifying each step) its analogue for approximation by algebraic polynomials of degree $\leqslant n$ on $[-1, 1]$:

$$E_n(f) \leqslant c\,\omega\Big(f, \frac{1}{n}\Big)\,.$$

c) Prove that the best approximation to $f_0(x) = \sqrt{1 - x^2}$ by algebraic polynomials of degree n satisfies

$$E_n(f_0) = \mathcal{O}(\tfrac{1}{n})\,.$$

Explain briefly why the inverse theorem, in the form given for trigonometric approximations, is not valid for the algebraic case.

5

1) Let $\Delta = (t_j)_{j=1}^{n+k}$ be a knot sequence such that $t_j < t_{j+k}$, and let $\mathcal{S}_k(\Delta)$ be the space of splines of degree $k-1$ spanned by the B-splines $(N_j)_{j=1}^n$ Let $\boldsymbol{x} = (x_i)_{i=1}^n$ be interpolation points obeying the conditions

$$N_i(x_i) > 0\,, \qquad 1 \leqslant i \leqslant n,$$

and let $P_{\boldsymbol{x}} : C[a,b] \to \mathcal{S}_k(\Delta)$ be the map which associates with any $f \in C[a,b]$ the spline $P_{\boldsymbol{x}}(f)$ from $\mathcal{S}_k$ which interpolates f at (x_i). Prove that

$$\frac{1}{d_k}\|A_{\boldsymbol{x}}^{-1}\|_{\ell_\infty} \leqslant \|P_{\boldsymbol{x}}\|_{L_\infty} \leqslant \|A_{\boldsymbol{x}}^{-1}\|_{\ell_\infty}\,,$$

where $A_{\boldsymbol{x}}$ is the matrix $(N_j(x_i))_{i,j=1}^n$, and d_k is the smallest constant such that

$$\frac{1}{d_k}\|a\|_{\ell_\infty} \leqslant \|\sum_{i=1}^n a_i N_i\|_{L_\infty} \qquad \forall\, a \in \mathbb{R}^n\,.$$

2) Consider the case of quadratic interpolating splines on the uniform knot-sequence $(t_1, t_2, \ldots, t_{n+3}) = (1, 2, \ldots, n+3)$ with the interpolating points

$$x_i = t_{i+2} = i+2, \quad i = 1, \ldots, n\,.$$

Prove that $\|P_{\boldsymbol{x}}\|_{L_\infty} = \mathcal{O}(n)$. Hence deduce that $P_{\boldsymbol{x}}$ is not bounded uniformly in n. (The inverse matrix $A_{\boldsymbol{x}}^{-1}$ should be found explicitly. You can use the fact that $d_3 = 3$).

6

(a) State the Chebyshev alternation theorem for the element of best uniform approximation to a function $f \in C[-1,1]$ from $\mathcal{P}_n$, the space of all algebraic polynomials of degree n. Prove that the algebraic polynomial of best approximation is unique.

(b) Let $T_n(x) = \cos n \arccos x$ be the Chebyshev polynomial of degree n, and let

$$f_0(x) = \sum_{k=0}^{\infty} a_k T_{3^k}(x), \quad \text{where} \quad a_k > 0, \quad \sum_{k=0}^{\infty} a_k < \infty, \quad x \in [-1,1].$$

Prove that, for every n, the polynomial p_n of best uniform approximation to f_0 is given by a partial sum of the series above, and determine the value $E_n(f_0) = \|f - p_n\|$ of best approximation in terms of a_k.

(c) Derive the lethargy theorem: for any monotone decreasing sequence $(\epsilon_n)_{n=0}^{\infty}$, with limit zero:

$$\epsilon_0 > \epsilon_1 > \cdots > \epsilon_n > \cdots, \qquad \epsilon_n \to 0,$$

there is a continuous function $f_\epsilon \in C[-1,1]$ such that

$$E_n(f_\epsilon) \geqslant \epsilon_n\,.$$

END OF PAPER