UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 5 June, 2013 9:00 am to 11:00 am

PAPER 60

DISTRIBUTION THEORY AND APPLICATIONS

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define the spaces $\mathcal{D}'(\mathbf{R})$ and $\mathcal{D}(\mathbf{R})$, specifying the notion of convergence in each.

For $u \in \mathcal{D}'(\mathbf{R})$ define the distributional derivative u' and show that it is an element of $\mathcal{D}'(\mathbf{R})$. For $h \in \mathbf{R}$ define $\tau_h u$, the translation of u by h, and show that

$$\lim_{h \to 0} \frac{\tau_{-h} u - u}{h} = u' \quad \text{in } \mathcal{D}'(\mathbf{R}).$$

Find the most general solution in $\mathcal{D}'(\mathbf{R})$ to the equations

$$\text{(a)}\ \ u_1' = 0, \qquad \text{(b)}\ \ x u_2 = 0.$$

Suppose $v \in \mathcal{D}'(\mathbf{R})$ satisfies the differential equation

$$-\frac{\mathrm{d}^n v}{\mathrm{d}x^n} + a_{n-1} \frac{\mathrm{d}^{n-1} v}{\mathrm{d}x^{n-1}} + \cdots + a_0 v = 0$$

in $\mathcal{D}'(\mathbf{R})$, where the $\{a_i\}_{i=0}^{n-1}$ are constants. Find the most general distributional solution to this equation and deduce that v is in fact a classical solution. Find the most general solution in $\mathcal{D}'(\mathbf{R})$ to the equation

$$-x \frac{\mathrm{d}^n u}{\mathrm{d}x^n} + a_{n-1} x \frac{\mathrm{d}^{n-1} u}{\mathrm{d}x^{n-1}} + \cdots + a_0 x u = 0.$$

2

State and prove the Malgrange–Ehrenpries theorem for non-zero constant coefficient partial differential operators. Your proof should involve the construction of a suitable "Hörmander staircase".

Let L be an ordinary differential operator with constant coefficients and order $N \geqslant 1$. Use your construction to show that L has a fundamental solution of the form $E = Hu$, where H is the Heaviside function and u is such that $Lu = 0$.

3

Write an essay on distributions defined via oscillatory integrals.

You should begin by defining the class of symbols $\mathrm{Sym}(X, \mathbf{R}^k; N)$ and what it means for a function $\Phi : X \times \mathbf{R}^k \to \mathbf{R}$ to be a phase function. For a symbol a and phase function Φ you should give meaning to

$$I_\Phi(a) = \int e^{i\Phi(x,\theta)} a(x,\theta) \, \mathrm{d}\theta$$

and prove that this gives rise to an element of $\mathcal{D}'(X)$ of finite order.

Finally, you should state a theorem relating the singular support of $I_\Phi(a)$ to the zero set of $\nabla_\theta \Phi(x,\theta)$ and discuss its application with the linear initial value problem

$$\frac{\partial u}{\partial t} + \mathbf{c} \cdot \nabla_{\mathbf{x}} u = 0, \qquad u(\mathbf{x}, 0) = \delta_0(\mathbf{x})$$

where $(\mathbf{x}, t) \in \mathbf{R}^n \times (0, \infty)$ and $\mathbf{c} \in \mathbf{R}^n$ is constant.

END OF PAPER