UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 31 May, 2013 1:30 pm to 4:30 pm

PAPER 6

TOPICS IN KINETIC THEORY

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

This questions concerns the linear transportation equation in $\mathbb{R}^3$

$$\begin{cases} \partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) + F(t,x,v) \cdot \nabla_v f(t.x.v) = 0 \\ f|_{t=0} = f_0, \end{cases}$$

where f, f_0 and F are in appropriate spaces.

(a) Write the characteristic equations for the linear transportation equation, and state conditions under which there exists a unique global solution to the characteristic equations.

(b) Sketch a short proof to the the existence and uniqueness theorem mentioned in (a).
Remark: Note that we only request a proof for the existence and uniqueness, not any regularity.

(c) Solve the characteristic equations for the linear transportation equation when $F(t,x,v) = v$, and use it to write an explicit solution to the linear transportation equation when $f_0 \in C^1_{x,v}\left(\mathbb{R}^3 \times \mathbb{R}^3\right)$.
Remark: You are not required to prove the formula for the explicit solution.

(d) Denoting by $X(t,x,v)$, $V(t,x,v)$ the solutions to the characteristic equations at time t, with initial datum $X(0,x,v) = x$ and $V(0,x,v) = v$ show that under the assumptions of (c) we have that

$$J(t,x,v) = \det\left(\frac{\partial(X,V)}{\partial(x,v)}\right) = e^{3t}.$$

(e) Show that in the general case, as in (a), one has that

$$\partial_t J(t,x,v) = \left(\nabla_V \cdot F(t,X,V)\right) J(t,x,v),$$

and conclude that if $\nabla_v \cdot F(t,x,v) = 0$ then $J(t,x,v) = 1$.
Remark: You may assume that all differentiations are allowed.

(f) Assume that the conditions of (a) are satisfied and $f_0 \in C^1_{x,v}\left(\mathbb{R}^3 \times \mathbb{R}^3\right)$. Show, using (e) and an expression to the general solution of the linear transportation equation, f_t, that $\|f_t(\cdot,\cdot)\|_{L^p(\mathbb{R}^3)}$ is conserved in time for any $1 \leqslant p < \infty$.

2

(a) Formally state Duhamel's Principle for the partial differential equation

$$\begin{cases} \partial_t f(t,y) + Df(t,y) = g(t,y) & t > 0,\ y \in Y. \\ f(0,y) = f_0(y) & y \in Y, \end{cases}$$

where D is a differential operator that has no time derivatives and Y is a domain without a boundary (this means you are required to give a solution that may only be valid formally).

(b) Use Duhamel's Principle to give an explicit solution to the equation

$$\begin{cases} \partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) + f(t,x,v) = g(t,x,v) & t > 0,\ x, v \in \mathbb{R}^d. \\ f|_{t=0} = f_0. \end{cases}$$

(c) Use Duhamel's Principle to write an expression for a possible solution to the linear Boltzmann equation

$$\begin{cases} \partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) + f(t,x,v) = Kf(t,x,v) & t > 0. \\ f|_{t=0} = f_0, \end{cases}$$

where $Kf(t,x,v) = \int_{\mathbb{R}^d} k(t,x,v,v_*) f(t,x,v_*) dv_*$. The solution should be expressed as a fixed point formula

$$f = F(f_0) + \tau(f)$$

where $F(f_0)$ is a function depending on f_0 alone and τ is an operator acting on the function f.

(d) We say that f is a weak solution to the linear Boltzmann equation in (c) if it solves the appropriate fixed point problem

$$f = F(f_0) + \tau(f).$$

Prove that if $k(t,x,v,v_*) = k_1(t,x,v) k_2(v_*)$ where k_1 continuous, bounded and non-negative, and k_2 is non-negative and in $L^1(\mathbb{R}^d)$ then there is a continuous and bounded weak solution, f_t, for all $t < T$, whenever f_0 is a continuous and bounded function.
Hint: Use the dominated convergence theorem to show that

$$\int_{\mathbb{R}^d} k_2(v_*) f(t,x,v_*) dv_*$$

is a well defined bounded continuous function when f is continuous and bounded.
Remark: Note that we only asked for the existence, not uniqueness.

(e) Assuming that the weak solution, if exists, is unique, show that if the continuous and bounded functions $f_{1,0}, f_{2,0}$ satisfy $f_{1,0}(x,v) \geqslant f_{2,0}(x,v)$ for all x, v then the weak solutions to the linear Boltzmann equation with initial datum $f_{1,0}$, $f_{2,0}$, described by the fixed point problem in (d), satisfy

$$f_{1,t} \geqslant f_{2,t},$$

for all time $t > 0$.

3

In this question we will consider the relaxation to equilibrium of the solution to the equation

$$\begin{cases} \partial_t f(t,v) = \rho(f)(t)M(v) - f(t,v) & v \in \mathbb{R}^d, t > 0. \\ f|_{t=0} = f_0, \end{cases}$$

where $\rho(f) = \int_{\mathbb{R}^d} f(v)dv$, $M(v)$ is a positive function with $\int_{\mathbb{R}^d} M(v)dv = 1$, $d \geqslant 1$, and f is in an appropriate space that would be described next.
We denote by $\mathcal{L}^2$ the space $L^2\left(\mathbb{R}^d, M^{-1}(v)dv\right)$, and define a linear operator $L : \mathcal{L}^2 \to \mathcal{L}^2$ by

$$L(f)(v) = \rho(f)M(v) - f(v).$$

With the above notation we can rewrite our problem as

$$\begin{cases} \partial_t f(t,v) = L(f)(t,v) & v \in \mathbb{R}^d, t > 0. \\ f|_{t=0} = f_0, \end{cases}$$

where $f, f_0 \in \mathcal{L}^2$.

(a) Show that $\rho(f)$ is well defined on $\mathcal{L}^2$ and that L is a bounded operator.

(b) Show that L is a self adjoint operator on $\mathcal{L}^2$ (i.e. $\langle Lf, g\rangle_{\mathcal{L}^2} = \langle f, Lg\rangle_{\mathcal{L}^2}$). Moreover, show that

$$\langle Lf, f\rangle_{\mathcal{L}^2} = -\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} \left(f(v_*)M(v) - f(v)M(v_*)\right)^2 M^{-1}(v)M^{-1}(v_*)dvdv_*$$

$$= -\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{f(v_*)}{M(v_*)} - \frac{f(v)}{M(v)}\right)^2 M^(v)M^(v_*)dvdv_* \leqslant 0.$$

In class we have used (b) and Jensen inequality for the probability measure $M(v)dv$ to show that

$$\langle L(f), f\rangle_{\mathcal{L}^2} \leqslant -\frac{1}{2}\left\|f - \rho(f)M\right\|^2_{\mathcal{L}^2},$$

from which we concluded that the spectral gap of the operator L is at least $\frac{1}{2}$, leading to

$$\left\|f_t - \rho(f_0)M\right\|_{\mathcal{L}^2} \leqslant e^{-\frac{t}{2}}\left\|f_0 - \rho(f_0)M\right\|_{\mathcal{L}^2}.$$

In what follows we will show that our estimation in class was too coarse, and we can actually get a better (and more accurate) spectral gap estimate.

(c) Define $\Pi : \mathcal{L}^2 \to \mathcal{L}^2$ by $\Pi(f) = \langle f, M\rangle_{\mathcal{L}^2} M$. Show that $\Pi = L + I$ and conclude that Π is linear, bounded and self adjoint.

(d) Show in addition that $\Pi^2 = \Pi$ and that $\mathrm{Ker}L = \mathrm{Im}\Pi$, proving that Π is an orthogonal projection on the kernel of L.

(e) Use (c) and (d) to show that

$$\langle L(f), f\rangle = \left\|\Pi(f)\right\|^2_{\mathcal{L}^2} - \left\|f\right\|^2_{\mathcal{L}^2} = -\left\|f - \Pi(f)\right\|^2_{\mathcal{L}^2}.$$

(f) Show that f_t is the solution to our differential equation with initial datum f_0 then $\Pi(f_t) = \Pi(f_0)$ for all $t > 0$, and use (e) to conclude that

$$\|f_t - \rho(f_0)M\|_{\mathcal{L}^2} = e^{-t}\,\|f_0 - \rho(f_0)M\|_{\mathcal{L}^2}\,.$$

Remark: Any solution that doesn't use (e) will not be awarded points.

4

In this problem we will consider a simple version of the spatially homogeneous Boltzmann equation on $\mathbb{R}^3$ and show convergence to equilibrium of its solution, under some conditions and a specific distance. The equation we will consider is

$$\begin{cases} \partial_t f(t,v) = Q(f,f)(t,v) \quad t>0 \\ f|_{t=0} = f_0, \end{cases}$$

where

$$Q(f,f) = \frac{1}{|\mathbb{S}^2|}\int_{\mathbb{R}^3\times\mathbb{S}^2} \big(f'(v)f'_*(v) - f(v)f(v_*)\big)\, dv_* d\sigma,$$

with $f'(v) = f(v')$, $f'_*(v) = f(v'_*)$, $f_*(v) = f(v_*)$ where

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma,$$

$\sigma \in \mathbb{S}^2$ and $d\sigma$ is the uniform surface measure on the unit sphere $\mathbb{S}^2$ (can be thought of as the restriction of the Lebesgue measure) with $\int d\sigma = |\mathbb{S}^2|$.
In what follows we will always assume that f is non negative

$$\int_{\mathbb{R}^3} f(v)dv = 1,$$

$$\int_{\mathbb{R}^3} vf(v)dv = 0$$

and

$$\int_{\mathbb{R}^3} |v|^2 f(v)dv = 1.$$

You may assume that f is smooth enough and decays nicely enough to allow all differentiations, integration by parts and Fubini Theorem.

(a) Defining the Fourier Transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}^3} f(v)e^{-i\xi\cdot v}dv$$

(notice that in class we had $-2\pi i\xi\cdot v$ in the exponent, instead of $-i\xi\cdot v$, this is just a different scaling), show that the first two conditions on f imply that

$$\left\|\widehat{f}\right\|_\infty \leqslant \widehat{f}(0) = 1, \quad \nabla\widehat{f}(0) = 0.$$

The goal of this exercise is to prove prove the so called Bobylev Identity: Taking the Fourier Transform in the v variable of our Boltzmann equation we get

$$\begin{cases} \partial_t \widehat{f}(\xi) = \frac{1}{|\mathbb{S}^2|}\int_{\mathbb{S}^2} \widehat{f}\left(\frac{\xi+|\xi|\sigma}{2}\right)\widehat{f}\left(\frac{\xi-|\xi|\sigma}{2}\right)d\sigma - \widehat{f}(\xi)\widehat{f}(0) \quad t>0 \\ \widehat{f}|_{t=0} = \widehat{f}_0. \end{cases}$$

(b) Use the fact that any orthogonal rotation R on $\mathbb{R}^3$ is a surjective map on the sphere $\mathbb{S}^2$ that preserves the volume, i.e.

$$\int_{\mathbb{S}^2} \phi(\sigma) d\sigma = \int_{\mathbb{S}^2} \phi(R(\sigma)) d\sigma$$

to prove that for any $x, y \in \mathbb{R}^3$

$$\int_{\mathbb{S}^2} \phi\left(x|y| \cdot \sigma\right) d\sigma = \int_{\mathbb{S}^2} \phi\left(y|x| \cdot \sigma\right) d\sigma,$$

when ϕ is continuous.
Hint: Use an appropriate rotation matrix connecting between the direction of x and y.

(c) Show that

$$\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} f(v')f(v'_*)e^{-iv\cdot\xi} dv dv_* d\sigma$$

$$= \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} f(v)f(v_*)e^{-i\frac{v+v_*}{2}\cdot\xi} e^{-i\frac{|v-v_*|}{2}\sigma\cdot\xi} dv dv_* d\sigma,$$

Hint: You may use the fact that the map

$$(v, v_*, \sigma) \to (v', v'_*, \sigma)$$

is surjective with Jacobian 1, as well as the formulas

$$v + v_* = v' + v'_*$$

and

$$v' - v'_* = |v - v_*|\,\sigma.$$

(d) Use (b) and (c) to show that

$$\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} f(v')f(v'_*)e^{-iv\cdot\xi} dv dv_* d\sigma = \int_{\mathbb{S}^2} \widehat{f}\left(\frac{\xi + |\xi|\sigma}{2}\right) \widehat{f}\left(\frac{\xi - |\xi|\sigma}{2}\right) d\sigma$$

Hint: You may also use the fact that the map

$$(v, v_*, \sigma) \to (v', v'_*, \sigma)$$

is surjective with Jacobian 1

With this identity at hand we can show convergence to equilibrium of the spatially homogeneous Boltzmann equation, under a specific form of distance.
item Let f, g be two non-negative function on $\mathbb{R}^3$ such that

$$\int_{\mathbb{R}^3} f(v) dv = \int_{\mathbb{R}^3} g(v) dv,$$

$$\int_{\mathbb{R}^3} v f(v) dv = \int_{\mathbb{R}^3} v g(v) dv$$

and

$$\int_{\mathbb{R}^3} |v|^2 f(v) dv, \int_{\mathbb{R}^3} |v|^2 f(v) dv < \infty.$$

It can be shown (you may take this as a given) that

$$d(f,g) = \sup_{\xi\in\mathbb{R}^3} \frac{\left|\widehat{f}(\xi) - \widehat{g}(\xi)\right|}{|\xi|^2}$$

is well defined metric, under the above conditions.

(e) Denoting by $\xi^{\pm} = \frac{\xi\pm|\xi|\sigma}{2}$ prove that $|\xi^-|^2 + |\xi^+|^2 = |\xi|^2$ and under the condition that $\int_{\mathbb{R}^3} f(v)dv = \int_{\mathbb{R}^3} g(v)dv = 1$

$$\frac{\left|\widehat{f}(\xi^+)\widehat{f}(\xi^-) - \widehat{g}(\xi^+)\widehat{g}(\xi^-)\right|}{|\xi|^2}$$

$$\leqslant \frac{\left|\widehat{f}(\xi^+) - \widehat{g}(\xi^+)\right|}{|\xi^+|^2}\cdot\frac{|\xi^+|^2}{|\xi|^2} + \frac{\left|\widehat{f}(\xi^-) - \widehat{g}(\xi^-)\right|}{|\xi^-|^2}\cdot\frac{|\xi^-|^2}{|\xi|^2}$$

$$\leqslant d(f,g).$$

(f) Use (a), (f) and Bobylev Identity to show that if f,g are solutions to the Boltzmann equation with the additional mentioned conditions then

$$\left|\partial_t\left(\frac{\widehat{f}(\xi) - \widehat{g}(\xi)}{|\xi|^2}\right) + \frac{\widehat{f}(\xi) - \widehat{g}(\xi)}{|\xi|^2}\right| \leqslant d(f,g)$$

Remark: The above is enough to show

$$d(f(t),g(t)) \leqslant d(f_0,g_0).$$

though we will not show it here.

5

(a) Let $A : \mathbb{R} \mapsto \mathbb{R}$ be C^∞ function. In general, we say that $u = u(y)$, $y \in \mathbb{R}^N$ solves

$$\mathrm{div}_y A(u) = g, \quad \text{in the sense of distribution,}$$

if and only if we have the following formulation

$$-\int_{\mathbb{R}^N} A(u(y)) \cdot \nabla_y \phi(y) \mathrm{d}y = \int_{\mathbb{R}^N} g(y)\phi(y)\mathrm{d}y, \tag{1}$$

for all function $\phi = \phi(y)$ that is C^∞ with respect to $y \in \mathbb{R}^N$ and has a compact support, i.e. the closure of $\{y \in \mathbb{R}^N : \phi(y) \neq 0\}$ is a compact subset of $\mathbb{R}^N$.

Suppose $f(t,x,v)$, $(t,x,v) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$ solves a linear transport equation

$$\partial_t f + v \cdot \nabla_x f = g, \quad x, v \in \mathbb{R}^3, \ t \in \mathbb{R}$$

in the sense of distribution. Write down the corresponding formulation like (1).

(b) Let $f(t,x,v), g(t,x,v) \in L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3)$, meaning that

$$\iiint_{\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3} f\phi \ \mathrm{d}x\mathrm{d}v\mathrm{d}t < \infty, \quad \iiint_{\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3} g\phi \ \mathrm{d}x\mathrm{d}v\mathrm{d}t < \infty,$$

for C^∞ function $\phi(t,x,v)$ with a compact support. Suppose that

$$\partial_t f + v \cdot \nabla_x f = g \quad \text{in a sense of distribution,}$$

and define $F(t,x,v) \equiv f(t, x+tv, v)$ and $G(t,x,v) \equiv g(t, x+tv, v)$.

Show that, for almost all x, v, as a function of $t \in \mathbb{R}$, $G(t,x,v) \in L^1_{\mathrm{loc}}(t \in \mathbb{R})$. [*Hint: You may use Fubini theorem and change of variables* $(t,x,v) \mapsto (t, x+tv, v)$.]

Show, for almost all x, v,

$$F(t_2,x,v) - F(t_1,x,v) = \int_{t_1}^{t_2} G(s,x,v)\mathrm{d}s, \quad \text{for all } t_1, t_2 \in \mathbb{R}.$$

[*Hint: You may use a test function* $\phi(t,x,v) = \phi_1(x-tv)\phi_2(t)$ *and proper change of variables. You may use the fact that if* $\int h(y)\rho(y)\mathrm{d}y = 0$ *for all* $\rho(y)$*, which is* C^∞ *and has compact support, then* $h(y) = 0$ *for almost all* y.]

(c) Let a linear functional L satisfies that, for all $h \in L^2(\mathbb{R}^3)$, (i.e. $\int_{\mathbb{R}^3} |h(v)|^2 \mathrm{d}v < \infty$),

$$\int_{\mathbb{R}^3} Lh(v)h(v)\mathrm{d}v \geqslant \delta \int_{\mathbb{R}^3} |h(v)|^2 \mathrm{d}v.$$

Suppose $f(t,x,v)$ is C^∞ and has a compact support and solves

$$\partial_t f + v \cdot \nabla_x f + Lf = 0, \quad \text{in the sense of distribution.}$$

Then prove that, for all $t_1 \leqslant t_2 \in \mathbb{R}$,

$$\left[\iint_{\mathbb{R}^3\times\mathbb{R}^3} |f(t_2,x,v)|^2 \mathrm{d}v\mathrm{d}x\right]^{1/2} \leqslant e^{-\delta(t_2-t_1)} \left[\iint_{\mathbb{R}^3\times\mathbb{R}^3} |f(t_1,x,v)|^2 \mathrm{d}v\mathrm{d}x\right]^{1/2}.$$

(d) The Boltzmann equation reads as

$$\partial_t f + v \cdot \nabla_x f = Q(f,f) \quad x, v \in \mathbb{R}^3.$$

where you can find the explicit form of $Q(f,f)$ in Problem 4. Assuming $f > 0$ solves the Boltzmann equation and f is C^∞ and has a compact support. Show that

$$\int_{\mathbb{R}^3} Q(f,f) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \mathrm{d}v = 0.$$

[Hint: You need some identities related with $\int_{\mathbb{R}^3} Q(f,f)(v)\phi(v)\mathrm{d}v$. Write down such identities clearly without proof.]

Also show that

$$\frac{d}{dt} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} f \log f \mathrm{d}v \mathrm{d}x \leqslant 0.$$

[Hint: You also need the fact $(1-X)\ln X \leqslant 0$ for all $X > 0$.]

(e) Write down explicitly one example of $f(t,x,v)$ which is not a constant function of x, but solves the Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = Q(f,f), \quad x, v \in \mathbb{R}^3.$$

Show that your example solves the equation by direct computation.

[*Hint: Use five parameters, for example a, b_1, b_2, b_3, c, to express the general solution $\mathcal{M}$ (Maxwellian) to $Q(\mathcal{M},\mathcal{M}) = 0$. Compute $\partial_t \mathcal{M} + v \cdot \nabla_x \mathcal{M} = 0$ and find the equations for a, b_1, b_2, b_3, c: you may compare the coefficients of polynomials of v. Find non-constant solutions of the equations.*]

END OF PAPER