UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 10 June, 2013 9:00 am to 11:00 am

PAPER 56

BINARY STARS

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A binary system consists of two stars of masses M_1 and M_2, total mass M, in a circular orbit of separation a. Show that the total orbital angular momentum has magnitude

$$J_{\text{orb}} = \frac{M_1 M_2}{M}\sqrt{GMa}.$$

In a simple model for the formation of a binary star, a spherical gas cloud of total mass M, instantaneous radius R, moment of inertia $\alpha M R^2$ and angular momentum J collapses homologously, rotating as a solid body and conserving angular momentum. Show that the ratio of centrifugal force to gravitational force for any fluid element varies proportional to $1/R$, where R is the radius of the cloud.

Collapse continues in this way until the centrifugal force at the equator balances the gravitational force there. At this point the cloud, then of radius R_{crit}, fissions into two equal mass spheres, each homologous to their parent. Assume that total angular momentum is conserved, that all material remains in corotation during this fission process and that, immediately afterwards, the two spheres are touching one another, in orbit about the centre of mass. Find the necessary condition on α for the orbital separation a to be less than R_{crit}.

The two daughter spheres then collapse further homologously, at constant separation, until they are again spinning so that the centrifugal force at the equator balances the gravity of the star there. At this point they fission again in an exactly analogous manner to produce four granddaughters orbiting one another in pairs separated by a'. Show that

$$\frac{a'}{a} = \frac{2\alpha^2}{(1+\alpha)^2}.$$

Comment on the viability of this model for the formation of multiple star systems.

2

A binary star consists of two point-like objects of masses M_1 and M_2 in an elliptical orbit with semi-major axis a and eccentricity e. From Newton's law of gravity show that the separation of the stars $\boldsymbol{r}$, evolves according to

$$\ddot{\boldsymbol{r}} = -\frac{GM}{r^3}\boldsymbol{r},$$

where $M = M_1 + M_2$ and $r = |\boldsymbol{r}|$.

In the absence of any perturbation show that the orbital energy E, angular momentum $\boldsymbol{J}$ and eccentricity e remain constant.

Show that at $\boldsymbol{x}$, far from the centre of mass of the binary star, its combined gravitational potential can be expanded to second order in its separation $\boldsymbol{r}$ as

$$\phi(\boldsymbol{x}) = -\frac{GM}{x} - Gq_{ij}l_{ij}(\boldsymbol{x}),$$

where

$$q_{ij} = \frac{\mu}{2}(3r_i r_j - r^2\delta_{ij}),$$

$$l_{ij}(\boldsymbol{x}) = \frac{3x_i x_j - x^2\delta_{ij}}{3x^5},$$

$\mu = M_1 M_2/M$, $r = |\boldsymbol{r}|$ and $x = |\boldsymbol{x}|$.

In the weak field limit for general relativity

$$\dot{E} = -\frac{4G}{45c^5}\frac{d^3 q_{ij}}{dt^3}\frac{d^3 q_{ij}}{dt^3}.$$

Show that, instantaneously,

$$\dot{E} = -\frac{32G^3M^2\mu^2}{5c^5r^4}\left(\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} - \frac{11}{12}\dot{r}^2\right),$$

where $\dot{r} = dr/dt$.

Explain why a circular orbit is expected to remain circular and show that, for such an orbit of separation a,

$$\frac{\dot{a}}{a} = -\frac{64G^3M^2\mu}{5c^5a^4}.$$

Comment on the consequences for the evolution of close binary stars.

3

A cataclysmic variable consists of a white dwarf of mass M_1 and a low-mass main-sequence star of mass M_2 in a circular orbit with separation a. The main-sequence star is filling its Roche lobe, of radius R_{L}, and transferring mass to the white dwarf at a rate $\dot{M}_1 = -\dot{M}_2$. The mass ratio $q = M_2/M_1 < 1$. The hydrostatic and thermal equilibrium radius of the main-sequence star can be approximated by

$$\frac{R_2}{R_\odot} = \frac{M_2}{M_\odot}$$

while, for a suitable range of mass ratios, the Roche-lobe radius R_{L} obeys

$$\frac{R_{\mathrm{L}}}{a} = 0.46 \left(\frac{M_2}{M}\right)^{\frac{1}{3}},$$

where $M = M_1 + M_2$. Show that the period P of the binary is given by

$$\frac{P}{P_0} = \frac{M_2}{M_\odot}$$

for some constant P_0.

Neglecting spin angular momentum find $\dot{R}_{\mathrm{L}}/R_{\mathrm{L}}$ as a function of $\dot{M}_2/M_2$ when $\dot{J} = 0$ and compare it with $\dot{R}_2/R_2$. Why would the mass transfer be dynamically unstable if $q > 4/3$?

Describe briefly the mechanisms that can lead to angular momentum loss $\dot{J} < 0$ and maintain mass transfer if $q < 4/3$.

In a classical nova, once a layer of hydrogen-rich material of mass $\delta m \approx 10^{-4}$ has accumulated on the surface of the white dwarf thermonuclear reactions ignite in the degenerate material. These expel the entire layer of mass δm isotropically from the system in the nova explosion over a time that is very short compared with the mass transfer timescale. Assume that the orbit remains circular and show that the change in separation $\delta a/a = \delta m/M$ and that the change in Roche lobe radius $\delta R_{\mathrm{L}}/R_{\mathrm{L}} = 4\delta m/3M$ to first order in $\delta m/M$.

Deduce that the mass transfer ceases. Assuming that there is a constant rate of angular momentum loss $-\dot{J}$ until mass transfer is resumed and that this rate remains the same until the next nova explosion show, again to first order in $\delta m/M$, that the ratio of the time spent detached t_{d}, during this interruption, to the time spent semi-detached t_{s}, while the hydrogen-rich layer is accumulating, is

$$\frac{t_{\mathrm{d}}}{t_{\mathrm{s}}} = \frac{2q}{(4-3q)(1+q)}.$$

END OF PAPER