UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 6 June, 2013 9:00 am to 11:00 am

PAPER 54

DYNAMICS OF ASTROPHYSICAL DISCS

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Viscous instability

The equation of mass conservation in cylindrical polar coordinates (r, φ, z) is given by:

$$\frac{\partial}{\partial t}(r\rho) + \frac{\partial}{\partial r}(r\rho u_r) + \frac{\partial}{\partial \varphi}(r\rho\Omega) + \frac{\partial}{\partial z}(r\rho u_z) = 0, \tag{1}$$

where ρ is the density and $\mathbf{u} = (u_r, r\Omega, u_z)$ the velocity. Angular momentum conservation is given by:

$$\frac{\partial}{\partial t}(r^3\rho\Omega) + \frac{\partial}{\partial r}(r^3\rho u_r\Omega - r^2 T_{r\varphi}) + \frac{\partial}{\partial \varphi}(r^3\rho\Omega^2 - rT_{\varphi\varphi}) + \frac{\partial}{\partial z}(r^3\rho u_z\Omega - r^2 T_{\varphi z}) = 0, \tag{2}$$

where T_{ij} denote components of the stress tensor, the most important of which is $T_{r\varphi} = \nu\rho r d\Omega/dr$, with ν the kinematic viscosity. In this question we consider the *viscous instability*, which may be relevant for some astrophysical discs.

(a) Show that equations (1) and (2) can be combined into a single diffusion equation for the surface density Σ in a Keplerian disc:

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(r^{1/2}\bar{\nu}\Sigma\right)\right], \tag{3}$$

where $\bar{\nu}$ is an appropriate average of the kinematic viscosity. Clearly state all assumptions you make.

(b) Find the general steady solution to equation (3). Use this to construct specific solutions for the cases of i) no viscous torque at the inner boundary r_{in}, and ii) no mass accretion onto the central object. Comment on what happens at the inner boundary in both cases.

(c) By considering perturbations $\Sigma_1(r,t)$ around a given solution to equation (3), $\Sigma_0(r,t)$, with $|\Sigma_1| \ll \Sigma_0$, show that

$$\frac{\partial \Sigma_1}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(r^{1/2}q\bar{\nu}_0\Sigma_1\right)\right], \tag{4}$$

where it is assumed that $\bar{\nu} = \bar{\nu}(r, \Sigma)$ with $q = \partial\ln(\bar{\nu}\Sigma)/\partial\ln\Sigma$ and $\bar{\nu}_0 = \bar{\nu}(r, \Sigma_0)$.

(d) Assume a viscosity law $\bar{\nu} \propto r^a\Sigma^b$, where a and b are constants. Derive a relation between a, b and p for a steady solution with no mass accretion (and therefore a constant viscous torque) found under part (b) with $\Sigma_0 \propto r^{-p}$, with p a constant. Show that, by a suitable change in variables, equation (4) can be transformed into the diffusion equation

$$\frac{\partial g}{\partial t} = A(x)\frac{\partial^2 g}{\partial x^2}, \tag{5}$$

and show that, for a specific choice of p, A is a constant. Adopting this value of p, show that for $q < 0$, the disc is unstable (viscous instability), and that the instability grows fastest at short wavelengths.

2 Gravitational instability with softening

The basic equations for a two-dimensional compressible inviscid shearing sheet with self-gravity read

$$\frac{\partial \Sigma}{\partial t} + \nabla \cdot (\Sigma \mathbf{u}) = 0, \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} = 2\Omega S x \mathbf{e}_x - \nabla \Phi_d - \nabla P/\Sigma, \tag{2}$$

where Σ is the surface density, $\mathbf{u}$ the two-dimensional velocity, $\boldsymbol{\Omega} = \Omega \mathbf{e}_z$ the rotation rate, S the shear rate, Φ_d the potential due to self-gravity and P the two-dimensional pressure, which we take to be a function of surface density only. The self-gravity potential can be found from Poisson's equation:

$$\nabla^2 \Phi_d = 4\pi G \rho = 4\pi G \Sigma \delta(z). \tag{3}$$

One major drawback of taking such a razor-thin disc is that the effects of self-gravity are exaggerated compared to a real disc with finite thickness. A possible solution is to introduce a softening length $\epsilon > 0$, which smooths the gravitational potential over a distance ϵ.

(a) Given a density distribution $\rho(x, y, z)$, an integral representation of Φ_d is given by

$$\Phi_d = -G \iiint \frac{\rho(x', y', z')dx'dy'dz'}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}.$$

Argue that, for the density distribution of a razor-thin disc, evaluating this potential at $z = \epsilon$ amounts to smoothing the horizontal gravitational forces over a distance ϵ. What would be an appropriate value for ϵ to represent a disc with finite thickness?

(b) Calculate the self-gravity potential from (3), and show that at $z = \epsilon$ its Fourier transform is given by (for $k \neq 0$):

$$\tilde{\Phi}_d = -\frac{2\pi G \tilde{\Sigma}}{k} \exp(-k\epsilon), \tag{4}$$

where $k = \sqrt{k_x^2 + k_y^2}$ and $\tilde{\Sigma}$ is the Fourier transform (in x and y) of the surface density.

(c) Consider a basic state with $\Sigma = \Sigma_0 = \text{cst}$, $P = P_0 = \text{cst}$ and $\mathbf{u} = \mathbf{u_0} = -Sx\mathbf{e}_y$. Consider axisymmetric perturbations $\propto \exp(\mathrm{i}k_x x - \mathrm{i}\omega t)$. Show that the dispersion relation is given by

$$\omega^2 = \kappa^2 - 2\pi G \Sigma_0 k \exp(-k\epsilon) + k^2 c_s^2, \tag{5}$$

where κ is the epicyclic frequency, and c_s is the sound speed in the unperturbed disc.

(d) Introduce a dimensionless wavenumber $s = Qc_s k/\kappa$, where the Toomre parameter $Q = \kappa c_s/(\pi G \Sigma_0)$, and a dimensionless smoothing length $\delta = \kappa\epsilon/(Qc_s)$. Assuming instability can occur for some values of s, write down a (non-algebraic) equation for the most unstable wavenumber s_*, and show that

$$s_* < \frac{1}{1+\delta}. \tag{6}$$

Derive an equation in terms of s_* for the minimum value of Q, Q_c, for which instability can occur, and show that

$$Q_c^2 < \frac{1+2\delta}{(1+\delta)^2} \leqslant 1. \quad (7)$$

Compare the results (6) and (7) to the unsoftened case ($\delta = 0$) and interpret the differences physically.

3 Satellite migration

Consider the dynamics of test particles around a satellite in the xy plane in the infinite shearing sheet. The equations of motion are given by

$$\ddot{x} - 2\Omega\dot{y} = 2\Omega S x - \frac{\partial \Psi}{\partial x}, \qquad (1)$$

$$\ddot{y} + 2\Omega\dot{x} = -\frac{\partial \Psi}{\partial y}, \qquad (2)$$

where $\Psi = -GM_s(x^2+y^2)^{-1/2}$ is the satellite potential, Ω is the fixed angular velocity of the satellite and S is the shear rate of the disc.

(a) Use the impulse approximation to show that the change in the velocity component parallel to the orbital motion for a particle starting at $x = x_0$ is given approximately by

$$\Delta v_{\parallel} \approx \frac{(GM_s)^2}{2S^3 x_0^5}. \qquad (3)$$

[You may want to make use of the integral $\int_0^\infty (1+s^2)^{-3/2} ds = 1$.]

(b) If a disc of test particles has surface density $\Sigma(x)$, show that, assuming a minimum impact parameter of H, and using (3), the torque on the disc for particles with $x > 0$ is given by

$$\Gamma_{>0} = r_0 \frac{(GM_s)^2}{2S^2} \int_H^\infty \frac{\Sigma(x) dx}{x^4}, \qquad (4)$$

where r_0 is the orbital radius of the satellite. Also show that when Σ is even around $x = 0$, the total torque on the disc is zero.

(c) Now take a surface density $\Sigma = \Sigma_0(1 + \beta x/r_0)$, where $\beta \ll r_0/H$ represents a measure of the local surface density slope. Show that the total torque on a Keplerian disc is given by

$$\Gamma = \frac{2}{9}\beta \frac{q^2}{(H/r_0)^2} \Sigma_0 r_0^4 \Omega^2, \qquad (5)$$

where $q = M_s/M_*$ is the mass ratio of the satellite and the central object. Discuss the flow of angular momentum in the system.

(d) Assuming that the satellite remains on a circular orbit, calculate its migration rate dr_0/dt due to disc torques.

END OF PAPER