UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 5 June, 2013 9:00 am to 11:00 am

PAPER 53

STRUCTURE AND EVOLUTION OF STARS

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

You may use the equations and results given below without proof.

The symbols used in these equations have the meanings that were given in lectures. Candidates are reminded of the equations of stellar structure in the form:

$$\frac{dm}{dr} = 4\pi r^2 \rho\,, \qquad \frac{dP}{dr} = -\,\frac{Gm\rho}{r^2}\,, \qquad \frac{dL_{\rm r}}{dr} = 4\pi r^2 \rho \epsilon\,.$$

In a radiative region

$$\frac{dT}{dr} = -\,\frac{3\kappa\rho L_{\rm r}}{16\pi a c r^2 T^3}\,.$$

In a convective region

$$\frac{dT}{dr} = \frac{(\Gamma_2 - 1)T}{\Gamma_2 P}\,\frac{dP}{dr}\,.$$

The luminosity, radius and effective temperature are related by $L = 4\pi R^2 \sigma T_{\rm e}^4\,.$

The equation of state for an ideal gas and radiation is $P = \dfrac{\mathcal{R}\rho T}{\mu} + \dfrac{aT^4}{3}\,,$ *with* $1/\mu = 2X + 3Y/4 + Z/2\,.$

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 a) A spherical star is composed of a gas with equation of state

$$P = K\rho^2,$$

which relates its pressure P to its density ρ for a constant K. Show that the radius of the star is

$$R = \left(\frac{K\pi}{2G}\right)^{\frac{1}{2}}$$

and that the ratio of its mean density $\bar{\rho}$ to its central density $\rho_{\rm c}$ is

$$\frac{\bar{\rho}}{\rho_{\rm c}} = \frac{3}{\pi^2}.$$

b) A cubic star of volume L^3 is composed of the same material. Show that it is possible to construct a solution to the structure equations for $0 \leqslant x \leqslant L$, $0 \leqslant y \leqslant L$ and $0 \leqslant z \leqslant L$ such that ρ vanishes on the faces of the cube. Show that, for this cubic star,

$$\frac{\bar{\rho}}{\rho_{\rm c}} = \frac{8}{\pi^3}.$$

c) Comment briefly on whether you expect such a star to exist in nature both theoretically and observationally.

2 a) The radiative transfer equation for intensity I_ν at frequency ν in a stellar atmosphere can be written as

$$\mu \frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = I_\nu(\mu, \tau_\nu) - S_\nu(\tau_\nu),$$

where τ_ν is the optical depth, μ is the cosine of the angle to the vertical and S_ν is a source function. Show that, when S_ν is linear in τ_ν, the emerging intensity $I_\nu(0, \mu)$ is linear in μ.

More generally S_ν may be written as a Taylor expansion so that

$$S_\nu = \sum_{j=0}^{n} a_j \tau_\nu^j.$$

Show that in this case

$$I_\nu = \sum_{j=0}^{n} A_j \mu^j$$

and determine the coefficients A_j.

When S_ν is independent of τ_ν show that the radial emergent intensity $I_\nu(1, 0)$ is related to the intensity and optical depth τ_ν by

$$I_\nu(1, 0) \approx I_\nu(1, \tau_\nu) + [S_\nu - I_\nu(1, \tau_\nu)]\tau_\nu,$$

when $\tau_\nu \ll 1$.

Describe the principle of absorption line formation in a stellar spectrum and discuss two situations in which we might expect to see emission lines in a star's spectrum.

b) You read in a paper that its authors have measured chemical abundances at the surface of a star of mass $M = 50\,M_\odot$ and luminosity $L = 10^6\,L_\odot$ from its observed spectrum. The authors state that they have used model atmospheres which assume a plane-parallel geometry, local thermodynamic equilibrium and a static atmosphere. Discuss whether these assumptions are valid.

3 In solar-like stars nuclear burning is dominated by the ppI and ppII chains

$$^1\mathrm{H}(^1\mathrm{H},\mathrm{e}^+\nu)^2\mathrm{H}(^1\mathrm{H},\gamma)^3\mathrm{He}(^3\mathrm{He},2^1\mathrm{H})^4\mathrm{He}$$

$$|$$

$$^3\mathrm{He}(^4\mathrm{He},\gamma)^7\mathrm{Be}(\mathrm{e}^-,\nu)^7\mathrm{Li}(^1\mathrm{H},^4\mathrm{He})^4\mathrm{He}.$$

The reaction rate between species i and j is

$$\frac{\lambda_{ij}n_i n_j}{1+\delta_{ij}}$$

where n_i is the number density of species i, δ_{ij} is the Kronecker delta and $\lambda_{ij} \propto \eta^2 \mathrm{e}^{-\eta}$ with $\eta = 42.48(AZ_i^2 Z_j^2 T_6^{-1})^{1/3}$, $A = A_i A_j/(A_i + A_j)$ is the reduced atomic mass of the two reacting nuclei, Z_i is the atomic number of species i and T_6 is related to temperature T by $T_6 = T/10^6\,\mathrm{K}$.

The beta decay of ^{7}Be is fast compared to all other reactions so that ^{7}Li is the predominant species of atomic mass 7 and all major species can be identified by $i \approx A_i$. Show that the rate r_{11} at the centre of the Sun, where $T_6 \approx 15$, of the reaction $^1\mathrm{H}(^1\mathrm{H},\mathrm{e}^+\nu)^2\mathrm{H}$ depends on temperature as $r_{11} \propto T^\alpha$, where $\alpha = \frac{1}{3}(\eta - 2) \approx 4$. Also show that β and γ are approximately 16 (with $\gamma > \beta$) in the expressions $r_{33} \propto T^\beta$ and $r_{34} \propto T^\gamma$.

Show that the rate of change of protons obeys

$$\frac{\mathrm{d}n_1}{\mathrm{d}t} = -\lambda_{11}n_1^2 - \lambda_{21}n_2 n_1 + \lambda_{33}n_3^2 - \lambda_{17}n_1 n_7,$$

and obtain the equivalent equations for n_2, n_3 and n_4.

At the centre of the Sun the characteristic timescale for $^1\mathrm{H}(^1\mathrm{H},\mathrm{e}^+\nu)^2\mathrm{H}$ is about 10^{10} yr while that of $^2\mathrm{H}(^1\mathrm{H},\gamma)^3\mathrm{He}$ is about 1 s. The characteristic timescale for n_3 to reach equilibrium is $\tau \approx 6 \times 10^5$ yr. By making an appropriate approximation, which you should explain, show that, near the centre of the Sun,

$$\frac{\mathrm{d}n_1}{\mathrm{d}t} \approx -\frac{3}{2}\lambda_{11}n_1^2 + \lambda_{33}n_3^2 - \lambda_{17}n_1 n_7$$

and

$$\frac{\mathrm{d}n_3}{\mathrm{d}t} \approx \frac{1}{2}\lambda_{11}n_1^2 - \lambda_{33}n_3^2 - \lambda_{34}n_3 n_4.$$

Show further that $n_3 \approx n_{3\mathrm{e}}$ where

$$n_{3\mathrm{e}} = -\frac{\lambda_{34}n_4}{2\lambda_{33}} + \sqrt{\left(\frac{\lambda_{34}n_4}{2\lambda_{33}}\right)^2 + \frac{\lambda_{11}n_1^2}{2\lambda_{33}}}.$$

Consider a small perturbation of the form $n_3 = n_{3\mathrm{e}} + x$ about this equilibrium and linearise the evolution equation for n_3 to obtain

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{x}{\tau},$$

where $\tau = (2\lambda_{33}n_{3\mathrm{e}} + \lambda_{34}n_4)^{-1}$.

Estimate the temperature at which τ is comparable to the age of the Sun.

Sketch the abundances X_1 and X_3 of ^{1}H and ^{3}He as a function of radius in the Sun today.

END OF PAPER