UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 31 May, 2013 1:30 pm to 4:30 pm

PAPER 52

ASTROPHYSICAL FLUID DYNAMICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

You are reminded of the equations of ideal magnetohydrodynamics in the form

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{u} \,,$$

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p = -\gamma p \nabla \cdot \mathbf{u} \,,$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\rho \nabla \Phi - \nabla p + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} \,,$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) \,,$$

$$\nabla^2 \Phi = 4\pi G \rho \,.$$

You may assume that for any vector $\mathbf{C}$

$$(\nabla \times \mathbf{C}) \times \mathbf{C} = \mathbf{C} \cdot \nabla \mathbf{C} - \frac{1}{2} \nabla (|\mathbf{C}|^2) \,,$$

for any vectors $\mathbf{C}$ *and* $\mathbf{D}$

$$\nabla \times (\mathbf{C} \times \mathbf{D}) = -\mathbf{D} \nabla \cdot \mathbf{C} + \mathbf{C} \nabla \cdot \mathbf{D} - \mathbf{C} \cdot \nabla \mathbf{D} + \mathbf{D} \cdot \nabla \mathbf{C} \quad ,$$

and that for $\mathbf{u} = (u_r, u_\theta, u_\phi)$ *in spherical polar coordinates* (r, θ, ϕ)

$$\nabla \cdot \mathbf{u} = \frac{1}{r^2} \frac{\partial (r^2 u_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (u_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} \,.$$

1

a) A stationary magnetized gas, governed by the ideal MHD equations for constant γ and without gravity, is such that ρ, p and $\mathbf{B}$ are constant. Suppose it undergoes linear adiabatic perturbation such that $p \to p + \delta p$, $\rho \to \rho + \delta\rho$ and $\mathbf{B} \to \mathbf{B} + \delta\mathbf{B}$. The quantities δp, $\delta\rho$ and $\delta\mathbf{B}$ are the perturbations to the corresponding state variables and the associated Lagrangian displacement is $\boldsymbol{\xi}$. Show that linearization of the ideal MHD equations gives

$$\delta\rho = -\rho\nabla\cdot\boldsymbol{\xi}, \quad \delta p = -\gamma p\nabla\cdot\boldsymbol{\xi}, \quad \delta\mathbf{B} = \mathbf{B}\cdot\nabla\boldsymbol{\xi} - \mathbf{B}\nabla\cdot\boldsymbol{\xi}, \qquad \text{and}$$

$$\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\nabla\left(\delta p + \frac{\delta\mathbf{B}\cdot\mathbf{B}}{\mu_0}\right) + \frac{\mathbf{B}\cdot\nabla\delta\mathbf{B}}{\mu_0}.$$

The space and time dependence of the perturbations is through a factor of the form $\exp(\mathrm{i}(\mathbf{k}\cdot\mathbf{r} - \omega t))$, where $\mathbf{k}$ is the wave vector and ω the wave angular frequency. Show that

$$\rho\omega^2\boldsymbol{\xi} = \mathbf{k}\left(\left(\gamma p + \frac{|\mathbf{B}|^2}{\mu_0}\right)(\mathbf{k}\cdot\boldsymbol{\xi}) - \frac{(\mathbf{B}\cdot\mathbf{k})(\mathbf{B}\cdot\boldsymbol{\xi})}{\mu_0}\right) + \frac{(\mathbf{B}\cdot\mathbf{k})((\mathbf{B}\cdot\mathbf{k})\boldsymbol{\xi} - \mathbf{B}(\mathbf{k}\cdot\boldsymbol{\xi}))}{\mu_0}$$

Show further that when $\mathbf{k}\cdot\boldsymbol{\xi} = 0$ and $\mathbf{B}\cdot\boldsymbol{\xi} = 0$, this equation admits solutions corresponding to Alfvén waves that are governed by the dispersion relation $\omega^2 = (\mathbf{k}\cdot\mathbf{v}_a)^2$ and give an expression for the phase velocity $\mathbf{v}_a$.

By finding a pair of simultaneous equations for $\mathbf{k}\cdot\boldsymbol{\xi}$ and $\mathbf{B}\cdot\boldsymbol{\xi}$, or otherwise, show that there are solutions corresponding to two additional waves whose phase speeds, v_p, are given by

$$v_p^2 = \frac{1}{2}(v_s^2 + v_a^2) \pm \sqrt{\left(\frac{1}{4}(v_s^2 + v_a^2)^2 - v_s^2 v_a^2\cos^2\theta\right)},$$

where $v_s = \sqrt{\gamma p/\rho}$ is the adiabatic sound speed, $v_a = |\mathbf{v}_a|$, and θ is the angle between $\mathbf{k}$ and $\mathbf{B}$. Briefly describe the properties of these waves.

b) Show that in Cartesian coordinates, (x, y, z), there are solutions of the full nonlinear ideal MHD equations for which ρ and p are constant and for which $\mathbf{u}$ and $\mathbf{B}$ are of the form

$$\mathbf{u} = (u_x(z,t), u_y(z,t), 0) \qquad \text{and} \qquad \mathbf{B} = (B_x(z,t), B_y(z,t), B_z),$$

where B_z is constant, provided that $B_x^2 + B_y^2$ is constant. Show also that

$$\frac{\partial^2 W}{\partial t^2} = \frac{B_z^2}{\mu_0\rho}\frac{\partial^2 W}{\partial z^2},$$

where W is either B_x or B_y. Suggest possible functional forms for B_x and B_y.

2

a) An unmagnetized ideal gas with constant γ is such that, adopting Cartesian coordinates (x, y, z), $\mathbf{u} = (0, 0, u(z,t))$ and p and ρ depend only on z and t. Gravity is negligible. Show that the equations governing the conservation of mass, momentum and energy may be expressed in the conservation law form

$$\frac{\partial \mathcal{U}}{\partial t} + \frac{\partial \mathcal{F}}{\partial z} = 0\,,$$

and give expressions for the quantities $\mathcal{U}$ and $\mathcal{F}$ in each case.

b) A stationary shock at $z = 0$ separates region 1 ($z > 0$, where $\rho = \rho_1$, $p = p_1$ and $u = u_1$) from region 2 ($z < 0$, where $\rho = \rho_2$, $p = p_2$ and $u = u_2$).

Use the statements of the conservation laws to derive the jump conditions (Rankine-Hugoniot relations) in the form

$$\begin{aligned} \rho_2 u_2 &= \rho_1 u_1\,, \\ p_2 + \rho_2 u_2^2 &= p_1 + \rho_1 u_1^2\,, \\ u_2(\rho_2 \frac{u_2^2}{2} + \gamma p_2/(\gamma - 1)) &= u_1(\rho_1 \frac{u_1^2}{2} + \gamma p_1/(\gamma - 1))\,. \end{aligned}$$

Use the above jump conditions to show that

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2 + 2} \quad \text{and} \quad \frac{p_2}{p_1} = \frac{2\gamma M_1^2 - (\gamma - 1)}{\gamma + 1},$$

where $M_1^2 = u_1^2/(\gamma p_1/\rho_1)$.

c) Now consider a shock front moving in the z direction with speed u_1 into a stationary medium with density ρ_1. Show that in the strong shock limit of large M_1, the post-shock density and pressure are $\rho_2 = ((\gamma+1)/(\gamma-1))\rho_1$ and $p_2 = 2\rho_1 u_1^2/(\gamma + 1)$, respectively.

A plane blast wave moving in the z direction into a stationary medium with $\rho = \rho_1$ and negligible pressure is generated by the instantaneous release of an energy per unit area, E, on the plane $z = 0$ at $t = 0$. The location of the shock front is given by $z = Z(t)$.

Assume that a similarity solution of the form

$$\rho = \rho_1 f(\eta), \quad p = \rho_1 \dot{Z}^2 g(\eta) \quad \text{and} \quad u = \dot{Z} h(\eta)$$

exists behind the shock for appropriate functions f, g and h, with the similarity variable $\eta = z/Z(t)$. Using the fact that the energy per unit area of the post-shock flow is conserved, show that the location of the shock is given by

$$Z(t) = C\left(\frac{E}{\rho_1}\right)^{1/3} t^{2/3},$$

where C is a constant.

3

A steady state axisymmetric isothermal magnetohydrodynamic flow under gravity is such that the magnetic field is purely poloidal and may be written in the form

$$\mathbf{B} = (B_R, 0, B_z) = -\frac{1}{R}\mathbf{e}_\phi \times \nabla\psi\,,$$

in cylindrical coordinates (R, ϕ, z). Here ψ is the magnetic flux function and $\mathbf{e}_\phi$ is the unit vector in the azimuthal direction. The velocity is of the form $\mathbf{u} = (u_R, 0, u_z)$. Show that

$$\mathbf{u} = \frac{k\mathbf{B}}{\rho}\,,$$

where ρ is the density and $k(\psi)$ is a function of ψ alone.

By considering the conservation of mass for a stream tube taking the form of a volume localised around a stream line with no inflow or outflow, and with infinitesimal cross sectional area A through which the interior flow is normal, deduce that $A \propto 1/|\mathbf{B}|$.

Show further from the equation of motion that

$$\frac{1}{2}|\mathbf{u}|^2 + \Phi + \ln(\rho)c_s^2 = \epsilon\,,$$

where c_s is the isothermal sound speed and $\epsilon(\psi)$ is a function of ψ alone. Deduce that

$$(c_s^2 - |\mathbf{u}|^2)\mathbf{B}\cdot\nabla\rho = -\rho\left[\mathbf{B}\cdot\nabla\Phi + |\mathbf{u}|^2\mathbf{B}\cdot\nabla(\ln(|\mathbf{B}|)\right]\,.$$

Accretion takes place through a slender column centred on the z axis, for $z \geqslant R_*$, onto the polar region of a central star of mass M_* and radius R_*. The normal cross-sectional area of the column where it joins the star is A_*. It is assumed that the radial component of the magnetic field is small enough that it can be neglected. Thus the motion can be taken to be in the z direction with $\mathbf{B} = (0, 0, B_z)$, where $B_z = B_* R_*^3/z^3$ and $\Phi = -GM_*/z$. Here B_* is the magnetic feld at the stellar surface.

Show that the flow has a critical point where $u = c_s$ and $z = z_s$, with $z_s = GM_*/(3c_s^2)$. If the flow at large z has a negligible speed with the density approaching $\rho = \rho_\infty$, show that the density at the critical point is $\rho_s = \exp(5/2)\rho_\infty$. Show further that the accretion rate onto the star is given by

$$\dot{M} = \rho_\infty c_s A_* \exp(5/2)\left(\frac{GM_*}{3c_s^2 R_*}\right)^3\,.$$

4

A non magnetic spherical star in hydrostatic equilibrium, such that in a spherical polar coordinate system (r, θ, ϕ) with origin at the centre of the star, $p = p(r)$ and $\rho = \rho(r)$, undergoes linear adiabatic perturbations such that $\rho \to \rho + \delta\rho(r,t)$ and $p \to p + \delta p(r,t)$, where $\delta\rho$ and δp are the perturbations to the density and pressure, respectively.

The adiabatic index, γ, varies with location in the star, thus $\gamma = \gamma(r)$. The velocity perturbation is $\mathbf{u} = (u(r,t), 0, 0)$ and the gravitational potential perturbation is $\delta\Phi(r,t)$. Show that these are connected by the relation

$$\frac{\partial^2(\delta\Phi)}{\partial r \partial t} = -4\pi G \rho u \,.$$

By linearizing the equation of motion, show that

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial\left(\gamma p \Delta u\right)}{\partial r} - \frac{4u}{r}\frac{dp}{dr}\,,$$

where $\Delta u = (1/r^2) d(r^2 u)/dr$.

If $u(r,t) = r\xi(r)\exp(-\mathrm{i}\omega t)$, show that $\xi(r)$ satisfies the equation

$$\omega^2 \rho r^4 \xi = -\frac{d}{dr}\left(\gamma p r^4 \frac{d\xi}{dr}\right) + \xi r^3 \frac{d}{dr}\left((4-3\gamma)p\right)\,.$$

Show that if p vanishes at the surface of the star, $r = R_s$, then ω^2 is an eigenvalue of a self-adjoint operator. Use this to write down a variational expression that can be used to determine whether the star is stable to radial perturbations. Show that the star will be unstable if

$$\int_0^{R_s} r^2 p (4 - 3\gamma) dr > 0.$$

END OF PAPER