UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 10 June, 2013 1:30 pm to 4:30 pm

PAPER 51

BLACK HOLES

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A particle of rest mass m with conserved energy $E = m$, falls inwards towards a Schwarzschild black hole of mass M. The particle moves in the equatorial plane $\theta = \frac{\pi}{2}$, with 4-velocity $u^\mu = \frac{dx^\mu}{d\tau}$, τ being proper time measured along its world line. Show that

$$u^\mu = \left(\frac{r}{r-2M}, -\frac{1}{r^{\frac{3}{2}}}\sqrt{2Mr^2 - h^2(r-2M)}, 0, \frac{h}{r^2}\right),$$

where h is the conserved angular momentum per unit mass.

Hence show that if the particle is to reach the horizon, then

$$|h| \leqslant 4M\,.$$

Two such particles moving in the same equatorial plane, and having equal rest masses, but different 4-velocities u_1^μ, u_2^μ, and angular momenta per unit mass h_1, h_2 collide at a radius r.

Assuming that their centre of mass energy $E_{\rm com}$ is given by

$$E^2_{\rm com} = -m^2 g^{\mu\nu}\left(u_1^\mu + u_2^\mu\right)\left(u_1^\nu + u_2^\nu\right),$$

show that

$$E^2_{\rm com} = 2m^2\left(1 - g_{\mu\nu}u_1^\mu u_2^\nu\right).$$

and hence that

$$E^2_{\rm com} = \frac{2m^2}{r^2(r-2M)} \times \Big[2r^2(r-M) - h_1h_2(r-2M) - \sqrt{2Mr^2 - h_1^2(r-2M)}\sqrt{2Mr^2 - h_2^2(r-2M)}\Big]\,. \qquad (1)$$

Show that the limit of the right hand side of (1) as $r \to 2M$ is

$$m^2\left(4 + \frac{(h_1-h_2)^2}{4M^2}\right).$$

Hence show that the centre of mass energy of the two particles at the horizon of the black hole can be no larger than $m\sqrt{20}$.

2

If K^μ is a Killing vector field, show that

$$2\nabla_\beta K_\gamma = \partial_\beta K_\gamma - \partial_\gamma K_\beta \,, \tag{1}$$

and

$$K_\alpha \big(\nabla^\beta K^\gamma\big)\big(\nabla_\beta K_\gamma\big) = 3\big(\nabla^\beta K^\gamma\big) K_{[\alpha} \nabla_\beta K_{\gamma]} - 2\big(\nabla_\sigma K_\alpha\big)\big(K^\tau \nabla_\tau K^\sigma\big)\,. \tag{2}$$

Deduce from (2) that if K^α has a Killing horizon with surface gravity κ then

$$\kappa^2 = -\frac{1}{2}\big(\nabla^\beta K^\gamma\big)\big(\nabla_\beta K_\gamma\big)\,.$$

Using both (1) and (2) calculate κ for a spherically symmetric metric of the form

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + r^2\big(d\theta^2 + \sin^2\theta d\phi^2\big)\,,$$

where the horizon is at $r = r_+$ at which $A(r_+) = 0$, $B(r_+) = 0\,, A'(r_+) \neq 0\,, B'(r_+) \neq 0$ and $'$ denotes differentiation with respect to r.

Hence show that the "Wick rotated" Riemannian metric

$$ds^2 = A(r)d\tau^2 + \frac{dr^2}{B(r)} + r^2\big(d\theta^2 + \sin^2\theta d\phi^2\big)\,,$$

will be free of a conical singularity at $r = r_+$ provided the coordinate τ is taken to have period $\frac{2\pi}{\kappa}$.

Indicate briefly the significance of this fact for the theory of Black Hole Thermodynamics.

3

The metric of a globally static asymptotically flat spacetime (with no horizon) may be cast in the form

$$ds^2 = -e^{2U(x^k)}dt^2 + e^{-2U(x^k)}\gamma_{ij}(x^k)(x^k)dx^i dx^j$$

with $i, j, k = 1, 2, 3$ and $U(x^k)$ a bounded function of the spatial coordinates x^k which tends to zero at infinity. The vacuum Einstein equations imply that

$$^{(3)}R_{ij} = 2\partial_i U \partial_j U \,,$$

where $^{(3)}R_{ij}$ is the Ricci tensor of the 3-metric γ_{ij}.

Show from the Bianchi identity for $^{(3)}R_{ij}$ that U satisfies $\gamma^{ij}\nabla_i\nabla_j U = 0$, where ∇_i is the Levi-Civita covariant derivative with respect to the metric γ_{ij}.

Show by multiplying $\gamma^{ij}\nabla_i\nabla_j U$ by U and integrating by parts, that if γ_{ij} is a complete non-singular Riemannian metric and there is no inner boundary, then $U = 0$ everywhere, and hence that $^{(3)}R_{ij} = 0$.

Given that in 3 dimensions the Riemann tensor $^{(3)}R_{ijpq}$ and Ricci tensor $^{(3)}R_{ij}$ are related by

$$^{(3)}R_{ijpq} = \gamma_{ip}\,{}^{(3)}S_{jq} - \gamma_{iq}\,{}^{(3)}S_{jp} - \gamma_{jp}\,{}^{(3)}S_{iq} + \gamma_{jq}\,{}^{(3)}S_{ip}$$

with $^{(3)}S_{iq} ={}^{(3)} R_{iq} - \frac{1}{4}\gamma_{iq}\gamma^{rs}\,{}^{(3)}R_{rs}$, show that the only asymptotically flat, non-singular globally static solution of the four-dimensional vacuum Einstein equations is Minkowski spacetime.

How does this statement change when horizons are allowed?

4

A Hermitian scalar quantum field $\hat{\Phi}$ has two expansions

$$\begin{aligned}\hat{\Phi} &= \left(\sum_i \hat{a}_i p_i + \hat{a}_i^\dagger \bar{p}_i\right) \\ &= \sum_i \left(\hat{a'}_i {p'}_i + \hat{a'}_i^\dagger \bar{p'}_i\right),\end{aligned}$$

where

$$\begin{aligned}[\hat{a}_i, \hat{a}_j^\dagger] &= [\hat{a'}_i, \hat{a'}_j^\dagger] = \delta_{ij} \\ [\hat{a}_i, \hat{a}_j] &= [\hat{a'}_i, \hat{a'}_j] = 0,\end{aligned}$$

and p_i, p'_i are appropriately normalized complex valued solutions of the covariant Klein-Gordon equation and $\bar{p}_i$ is the complex conjugate of p_i and $\bar{p'_i}$ is the complex conjugate of ${p'}_i$. Assuming

$$\hat{a'}_i = \sum_j \left(\alpha_{ij}\hat{a}_j + \beta_{ij}\hat{a}_j^\dagger\right),$$

give conditions on α_{ij} and β_{ij} ensuring that $\hat{a'}_i$ and $\hat{a'}_i^\dagger$ satisfy the canonical commutation relations, given that $\hat{a}_i$ and $\hat{a}_i^\dagger$ satisfy the canonical commutation relations. If

$$\hat{a}_i = \sum_j \left(A_{ij}\hat{a'}_j + B_{ij}\hat{a'}_j^\dagger\right),$$

give expressions for A_{ij} and B_{ij} in terms of α_{ij} and β_{ij}.

Explain how α_{ij} and β_{ij} may be obtained from the relation between the solutions p_i and p'_i .

Given that the system is in the state $|0\rangle$ such that $\hat{a}_i|0\rangle = 0$, calculate $\langle 0|\hat{a'}_i^\dagger \hat{a'}_i|0\rangle$ and give its interpretation.

Show that if $\hat{a'}_i|0'\rangle = 0$, then $|0\rangle$ is some multiple of $e^{\hat{F}}|0'\rangle$, where

$$\hat{F} = \frac{1}{2}\sum_i \sum_j M_{ij}\hat{a'}_i^\dagger \hat{a'}_j^\dagger$$

and M_{ij} should be given in terms of of A_{ij} and B_{ij}.

Illustrate your results by giving a brief sketch of Hawking's derivation of black hole radiation.

END OF PAPER