UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 7 June, 2013 1:30 pm to 3:30 pm

PAPER 50

APPLICATIONS OF DIFFERENTIAL GEOMETRY TO PHYSICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

By consideration of the map $\phi : \mathbb{R}^3/\mathbb{R}^2 \to \mathbb{C}$ given by

$$\phi(x, y, z) = \frac{x + iy}{1 - z}, \quad z \neq 1$$

show that a two–dimensional sphere S^2 admits an atlas with holomorphic transition functions.

Let $f : S^2 \to S^2$ be given by $f(\zeta) = \zeta^k$, where $\zeta \in \mathbb{C} \cup \{\infty\}$, and k is a non-negative integer. Use the volume form

$$\Omega = \frac{i d\zeta \wedge d\bar{\zeta}}{(1 + |\zeta|^2)^2}$$

on S^2 to compute the topological degree of f. Compare your answer with the method of pre-images.

2

Write an essay on the interplay between gauge theory, and the theory of connections on $SU(2)$ principal bundles.

3

Let G be a two–dimensional Lie group acting on $\mathbb{R}$ by

$$g(\alpha, \beta; x) = \exp(\alpha)x + \beta,$$

where $(\alpha, \beta) \in \mathbb{R}^2$ are the parameters of the transformation, and $x \in \mathbb{R}$.

- Consider vectors in $\mathbb{R}^2$ of the form $(x, 1)$ to show that the elements of G can be represented by matrices
$$g = \begin{pmatrix} \exp(\alpha) & \beta \\ 0 & 1 \end{pmatrix},$$
and use this representation to find the Lie algebra of G.
- Use the Maurer–Cartan one–form to find left and right invariant vector fields on G, and show that they satisfy the Lie algebra relations.
- Consider the elements of G
$$g_1 = \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} \exp(\epsilon) & 0 \\ 0 & 1 \end{pmatrix}$$
to show that left translations on G are generated by right-invariant vector fields.

4

Let (M, g) be an n–dimensional Riemannian manifold.

- Consider the canonical symplectic structure on T^*M to show that the geodesics on M are integral curves of a Hamiltonian vector field on T^*M with the Hamiltonian
$$H = \frac{1}{2} g^{ij}(x) p_i p_j, \quad i, j = 1, \ldots, n$$
where $x \in M$.
- Show that homogeneous quadratic polynomials of the form $K = K^{ij}(x) p_i p_j$ Poisson commute with H iff $K_{ij} = g_{ik} g_{jl} K^{kl}$ is a Killing tensor on M.
- Let Y be a two–form on M such that
$$\nabla_{(i} Y_{j)k} = 0,$$
where ∇ is the Levi–Civita connection of g. Show that $K_{ij} = Y_{ik} {Y_j}^k$ is a Killing tensor.

END OF PAPER