UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 3 June, 2013 1:30 pm to 4:30 pm

PAPER 5

REPRESENTATION THEORY

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

What does it mean for a finite-dimensional $\mathbb{C}$-algebra A to be semisimple? Assume A is semisimple. State and prove the Artin–Wedderburn theorem for A [basic facts about such algebras may be assumed, provided they are clearly stated.]

Deduce the following results:

(a) Let B be any finite-dimensional $\mathbb{C}$-algebra and let S be a B-module. Show that the *structural algebra homomorphism* $B \to \mathrm{End}_{\mathbb{C}}(S)$ defined by

$$a \to (s \mapsto as)$$

is surjective if and only if S is an irreducible B-module.

(b) Let $C = \mathbb{C}G$, the ordinary group algebra of a finite group. Prove that the number of isomorphism classes of irreducible C-modules is equal to the number of conjugacy classes in G. [You may assume the result that the centre $Z(C)$ of C is free as a $\mathbb{C}$-module with rank equal to the number of conjugacy classes in G.]

2

Let $m \in \mathbb{N}$. Given $x = (x_1, \ldots, x_m) \in \mathbb{C}^m$ and $\ell_1, \ldots, \ell_m \in \mathbb{Z}$, define $|x^{\ell_1}, \ldots, x^{\ell_m}|$. Define also the ith power sum $s_i(x)$ for $i \in \mathbb{N}$. Given a conjugacy class for S_n show that, in the usual notation,

$$s_1^{\alpha_1} \ldots s_n^{\alpha_n} |x^{m-1}, \ldots, 1| = \sum \omega_\lambda(\alpha) |x^{\ell_1}, \ldots, x^{\ell_m}|$$

where the sum is taken over all partitions λ of n into at most m parts, and ω_λ are certain class functions to be explicitly defined.

Assume now that $\omega_\lambda = \chi^\lambda$, the character of the Specht module indexed by λ.

(a) Use the above character formula to compute $\chi^{(2,1)}$ for the conjugacy class of transpositions in S_3. Check that this is the same as the value $\chi_V(12)$, where V is the standard representation $V = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 0\}$ with character χ_V.

(b) Let λ be a partition of n. Let M^λ be the permutation module on cosets of Young subgroups of S_n. Let σ lie in the conjugacy class indexed by μ and let m_q be the multiplicity with which the integer q occurs in μ. By considering the polynomial

$$p_\mu(x_1, \ldots, x_n) = \prod_{q=1}^{n} (x_1^q + \cdots + x_n^q)^{m_q}$$

or otherwise, find an expression for the value of the character of M^λ on elements lying in the class indexed by μ.

3

Let t_λ be a λ-tableau. Define the Young symmetrizer $h(t_\lambda)$. If V is an m-dimensional vector space and $n \in \mathbb{N}$, write down the actions of S_n and $\mathrm{GL}(V)$ on the tensor space $V^{\otimes n}$ and show that they commute. Prove there is a decomposition

$$V^{\otimes n} = \bigoplus h(t_\lambda) V^{\otimes n} ,$$

where the sum is over all partitions of n and standard t_λ [combinatorial properties of Young symmetrizers can be assumed, as can a certain decomposition of $\mathbb{C}S_n$, provided they are clearly stated.]

Show that the non-zero modules of the form $h_\lambda V^{\otimes n}$ for λ running through the partitions of n are the non-isomorphic irreducible $\mathbb{C}\mathrm{GL}(V)$-modules [standard results about the Schur algebra may be quoted].

State conditions on λ for the modules $h_\lambda V^{\otimes n}$ to be non-zero.

What does it mean to say that a finite-dimensional $\mathbb{C}\mathrm{GL}(V)$-module is rational? Write down the 1-dimensional rational representations of $\mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})$. Hence, or otherwise, prove that every 1-dimensional rational representation of $\mathrm{GL}(V)$ is of the form $\det^r : \mathrm{GL}(V) \to \mathrm{GL}_1$ where $r \in \mathbb{Z}$ [if you wish, you may assume that the diagonalisable matrices are Zariski-dense in GL_n.]

Finally show that every rational representation ρ of $\mathbb{C}\mathrm{SL}(V)$ is completely reducible and that ρ is the restriction of a rational representation ρ' of $\mathbb{C}\mathrm{GL}(V)$, and ρ is irreducible if and only if ρ' is irreducible.

4

Let V be an m-dimensional vector space over $\mathbb{C}$. For the m-tuple $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}^m$ with $\lambda_1 \geqslant \dots \geqslant \lambda_m$, denote the character of $D_{\lambda_1,\cdots,\lambda_m}(V)$ by ϕ_λ. Show that such modules are a complete set of the non-isomorphic irreducible rational $\mathbb{C}\mathrm{GL}(V)$-modules.

Show also that if $\xi \in \mathrm{End}(V)$ then $\phi_\lambda(\xi)$ is a symmetric function of the eigenvalues of ξ.

Assuming Weyl's character formula, state and prove a formula for the degree, $\deg\phi_\lambda$, of ϕ_λ. Deduce that if two rational $\mathbb{C}\mathrm{GL}(V)$-modules have the same character then they are isomorphic.

Finally show that each polynomial representation of $\mathrm{GL}(V)$ occurs exactly once in in the symmetric power

$$\bigoplus_j \mathrm{S}^j(V \oplus \Lambda^2 V)\,.$$

[For the final part you may assume an identity of Schur:

$$\prod_{i=1}^{m}(1-x_i)^{-1} \cdot \prod_{1 \leqslant i < j \leqslant m}(1-x_i x_j)^{-1} = \sum_\lambda s_\lambda(x_1, \dots, x_m)\,,$$

where $s_\lambda(x)$ is the Schur polynomial.]

5

Define the terms *hook*, *hook length* and *hook graph.*

Let $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_m \geqslant 0)$, and set $\ell_i = \lambda_i + m - i$, a β-set for λ. State the hook-length formula for the dimension of the Specht module. Show that the hook-length formula is equivalent to the formula

$$f_\lambda := n!\,\frac{\prod_{i<j}(\ell_i - \ell_j)}{\ell_1!\ell_2!\ldots\ell_m!}. \tag{1}$$

Show that $f_\lambda = \sum_{i=1}^m f_{(\lambda_1,\ldots,\lambda_i-1,\ldots,\lambda_m)}$, where $f_{(\lambda_1,\ldots,\lambda_i-1,\ldots,\lambda_m)}$ is defined to be zero if the sequence is not weakly decreasing, i.e. if $\lambda_i = \lambda_{i+1}$. Deduce an inductive proof of the hook-length formula by showing that if $F(\ell_1,\ldots,\ell_m)$ is the expression on the right-hand side of (1), then

$$F(\ell_1,\ldots,\ell_m) = \sum_{i=1}^m F(\ell_1,\ldots,\ell_i - 1,\ldots,\ell_m)\,.$$

Show that this is equivalent to the formula

$$n\Delta(\ell_1,\ldots,\ell_m) = \sum_{i=1}^m \ell_i \cdot \Delta(\ell_1,\ldots,\ell_i - 1,\ldots,\ell_m)\,,$$

where we write $\Delta(\ell_1,\ldots,\ell_m)$ for $\prod_{i<j}(\ell_i - \ell_j)$. Deduce this formula from the identity

$$\sum_{i=1}^m x_i\Delta(x_1,\ldots,x_i + t,\ldots,x_m) = (x_1 + \cdots + x_m + \binom{m}{2}t)\cdot\Delta(x_1,\ldots,x_m)\,.$$

Prove this identity.

Finally deduce the identity, that for any $m \geqslant 2$,

$$\sum \frac{\prod_{i<j}(\ell_i - \ell_j)^2}{\ell_1!^2\ell_2!^2\cdots\ell_m!^2} = 1\,,$$

where the sum is over all m-tuples $\ell_1,\ldots,\ell_m$ of non-negative integers whose sum is $(m+1)m/2$.

END OF PAPER