UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 3 June, 2013 9:00 am to 12:00 pm

PAPER 49

GENERAL RELATIVITY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let T be a tensor field of type (r,s) and X a vector field. Define the Lie derivative $\mathcal{L}_X T$.

(b) Prove that the Lie derivative commutes with contraction and satisfies the Leibnitz rule

$$\mathcal{L}_X(S \otimes T) = (\mathcal{L}_X S) \otimes T + S \otimes (\mathcal{L}_X T)$$

where S is a tensor field of type (p,q). [*You may assume the existence of coordinates* $(t, x^1, x^2, \ldots)$ *such that* $X = \partial/\partial t$]

(c) Prove that $\mathcal{L}_X f = X(f)$ and $\mathcal{L}_X Y = [X,Y]$ where f is a function and Y a vector field.

(d) Let ω be a covector field. Prove that, in any coordinate basis,

$$(\mathcal{L}_X \omega)_\mu = X^\nu \omega_{\mu,\nu} + \omega_\nu X^\nu{}_{,\mu}$$

(e) Let T be tensor field of type $(1,1)$. Derive an expression for the components of $\mathcal{L}_X T$ in an arbitrary coordinate basis. Hence or otherwise prove that

$$\mathcal{L}_X \mathcal{L}_Y T - \mathcal{L}_Y \mathcal{L}_X T = \mathcal{L}_{[X,Y]} T$$

2

In the linearized approximation to General Relativity it is assumed that the spacetime manifold is $\mathbb{R}^4$ and there exist "almost inertial" coordinates $x^\mu = (t, x^i)$ such that the metric takes the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $\eta_{\mu\nu} = \text{diag}(-1,1,1,1)$ and $|h_{\mu\nu}| \ll 1$. Indices are raised with $\eta^{\mu\nu}$ and lowered with $\eta_{\mu\nu}$. Let $\bar{h}_{\mu\nu} = h_{\mu\nu} - (1/2)h\eta_{\mu\nu}$ where $h = h^\rho{}_\rho$. Then, in the gauge $\partial^\nu \bar{h}_{\mu\nu} = 0$, the linearized Einstein equation is $\partial^\rho \partial_\rho \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$.

(a) Assume that matter is confined within a ball of radius d centred on the origin. Assume that the matter moves non-relativistically. Show that, for $r = |\mathbf{x}| \gg d$

$$\bar{h}_{ij}(t, \mathbf{x}) \approx \frac{2}{r}\ddot{I}_{ij}(t - r)$$

where

$$I_{ij}(t) = \int d^3x \, T_{00}(t, \mathbf{x}) x^i x^j$$

(b) Consider a binary system consisting of two stars, each of mass M, in a Newtonian circular orbit of radius R. Assuming that the quadrupole formula applies to this system, calculate the average power emitted in gravitational radiation. You may approximate the energy-momentum tensor of a star at rest at the origin as $T_{00} = M\delta^3(\mathbf{x})$, $T_{0i} = T_{ij} = 0$. [*Hint: Take the positions of the stars to be* $\pm(R\cos\Omega t, R\sin\Omega t, 0)$ *where* Ω *is to be determined.*]

(c) Explain why the binary systems which emit the most gravitational radiation are tightly bound and involve neutron stars or black holes.

3

(a) Consider an infinitesimal variation of the spacetime metric $g_{ab} \to g_{ab} + \delta g_{ab}$.

(i) Show that the corresponding change in the Levi-Civita connection is given by

$$\delta\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(\nabla_c \delta g_{db} + \nabla_b \delta g_{dc} - \nabla_d \delta g_{bc}\right)$$

where ∇ is the Levi-Civita connection associated to g_{ab}.

(ii) Show that the change in the Ricci tensor is

$$\delta R_{ab} = \nabla_c \delta\Gamma^c_{ab} - \nabla_b \delta\Gamma^c_{ac}$$

[*You may use the formula for the components of the Riemann tensor in a coordinate basis:*

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\tau_{\nu\sigma}\Gamma^\mu_{\tau\rho} - \Gamma^\tau_{\nu\rho}\Gamma^\mu_{\tau\sigma} \quad]$$

(iii) Show that the change in the Ricci scalar is

$$\delta R = -R^{ab}\delta g_{ab} + \nabla^a \nabla^b \delta g_{ab} - \nabla^c \nabla_c \left(g^{ab}\delta g_{ab}\right)$$

(b) A theory of gravity has action

$$S = \int d^4x \sqrt{-g}\, f(R) + S_{\text{matter}}$$

where f is a smooth function.

(i) Show that varying g_{ab} gives the equation of motion $E_{ab} = (1/2)T_{ab}$ where T_{ab} is the energy-momentum tensor of matter and

$$E_{ab} = f' R_{ab} - f'' \nabla_a \nabla_b R - f''' \nabla_a R \nabla_b R + \left(-\frac{1}{2} f + f'' \nabla^c \nabla_c R + f''' \nabla^c R \nabla_c R\right) g_{ab}$$

where f' denotes $f'(R)$ etc. [*You may use without proof* $\delta g = g g^{ab} \delta g_{ab}$]

(ii) Explain why $\nabla^a E_{ab} = 0$ for any metric g_{ab}. [*Hint. This does not require a long calculation.*]

(iii) Why does Lovelock's theorem not apply to this theory?

(iv) Determine the necessary and sufficient condition on the function f that ensures that any solution of the vacuum Einstein equation with cosmological constant Λ is also a vacuum solution of this theory.

4

A spacetime has metric

$$ds^2 = \frac{1}{z^2}\left(-f(z)^2 dt^2 + dx^2 + dy^2 + f(z)^{-2} dz^2\right)$$

where $f(z) = \sqrt{1 - \alpha z^3}$ and α is a constant. An orthonormal basis is defined by

$$e^0 = \frac{f}{z} dt \qquad e^1 = \frac{1}{z} dx \qquad e^2 = \frac{1}{z} dy \qquad e^3 = \frac{1}{zf} dz$$

(a) Determine the connection 1-forms satisfying $de^\mu = -\omega^\mu{}_\nu \wedge e^\nu$ and the curvature 2-forms defined by $\Theta^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu$

(b) Show that this metric satisfies the vacuum Einstein equation with a cosmological constant Λ whose value you should determine.

END OF PAPER