UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 5 June, 2013 1:30 pm to 4:30 pm

PAPER 48

COSMOLOGY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Consider an FRW universe with a cosmological constant, spatial curvature and a perfect fluid with density ρ and constant equation of state $w \equiv P/\rho \geqslant 0$. Use units where $8\pi G = c = 1$.

Show that the Friedmann equation can be written as the equation of motion of a particle moving in one dimension with vanishing total energy and potential

$$V(a) = -\frac{\rho_0}{6}\frac{1}{a^{(1+3w)}} + \frac{k}{2} - \frac{\Lambda}{6}a^2 .$$

Sketch $V(a)$ for the following cases: *i)* $k = 0$, $\Lambda < 0$, *ii)* $k = \pm 1$, $\Lambda = 0$, and *iii)* $k = 0$, $\Lambda > 0$. Assuming that the universe "starts" with $da/dt > 0$ near $a = 0$, describe the evolution in each case. Where applicable determine the maximal value of the scale factor.

(b) From now on consider the case $\Lambda = 0$ and $k = +1$.

Show that the scale factor obeys the following differential equation

$$a'' + a = \frac{\rho_0}{6}(1-3w)a^{-3w} ,$$

where the primes denotes derivatives with respect to conformal time τ. You may assume that this equation has the following solution [*do not try to show this!*]

$$a(\tau) = A\left[\sin\left(\frac{1+3w}{2}\tau + B\right)\right]^{2/(1+3w)} ,$$

where A and B are integration constants.

On physical grounds determine the constant A in terms of ρ_0 and w.

Defining $a(\tau = 0) \equiv 0$, give the solution for *i)* pressureless dust ($w = 0$) and *ii)* radiation ($w = \frac{1}{3}$). In each case, determine the time of the "big crunch".

Consider a photon leaving the origin at $\tau = 0$. For the case of pressureless dust, how many times can the photon circle the universe before the universe "ends"? How far does the photon get in the case of radiation?

[*Hint: You may use that the metric for a closed FRW universe is*

$$\mathrm{d}s^2 = a^2(\tau)\left[\mathrm{d}\tau^2 - \mathrm{d}\chi^2 - \sin^2\chi\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] . \quad]$$

2

The early universe was filled with a gas of photons, electrons and protons. (You may ignore other light nuclei such as helium.) Around 380,000 years after the Big Bang, electrons and protons combined into neutral hydrogen atoms. The number density of massive particles in thermal equilibrium at a temperature $T \ll m_i$ is

$$n_i = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-(m_i-\mu_i)/T} \ ,$$

in units where $\hbar = c = k_B \equiv 1$, while the number density of photons is

$$n_\gamma = \frac{2\zeta(3)}{\pi^2} T^3 \ , \qquad \text{where} \quad \zeta(3) \approx 1.2 \ .$$

a) Derive the Saha equation for the fractional ionization $X_e = n_e/n_b$,

$$\frac{1-X_e}{X_e^2} = \frac{2\zeta(3)}{\pi^2}\,\eta \left(\frac{2\pi T}{m_e}\right)^{3/2} e^{\mathcal{B}/T} \ = \ 3\times 10^{-16} \frac{e^{\mathcal{B}/T}}{(\mathcal{B}/T)^{3/2}} \ ,$$

where $\eta = n_b/n_\gamma \approx 6\times 10^{-10}$ is the baryon-to-photon ratio, $\mathcal{B} = 13.6$ eV is the binding energy of hydrogen, and $m_e = 511$ keV is the electron mass.

b) Defining "recombination" as the moment when $X_e(T_{\rm rec}) \equiv \frac{1}{2}$, show that

$$T_{\rm rec} \sim \frac{2}{5}\,\text{eV} \ .$$

[*Hint: You may use that* $\ln\left(\frac{2}{3}\times 10^{16}\right) \approx 36$.]

Explain why $T_{\rm rec}$ is much smaller than $\mathcal{B}$.

c) Qualitatively discuss "photon decoupling" and "electron freeze-out". Define the decoupling temperature $T_{\rm dec}$. Sketch the expected $X_e(T)$ together with the prediction of the Saha equation. Mark $T_{\rm rec}$ and $T_{\rm dec}$ on the plot.

3

The universe contains radiation (r), pressureless matter (m) and dark energy (Λ). Inside the Hubble-radius, the matter fluctuations satisfy the following equations

$$\dot{\delta}_m = -\frac{1}{a}\boldsymbol{\nabla}\cdot\boldsymbol{v}_m \quad \text{and} \quad \dot{\boldsymbol{v}}_m = -\frac{\dot{a}}{a}\boldsymbol{v}_m - \frac{1}{a}\boldsymbol{\nabla}\Phi\ ,$$

where δ_m and $\boldsymbol{v}_m$ are the density contrast and peculiar velocity of the matter. Overdots denote derivatives with respect to proper time t. The gravitational potential Φ is determined by the Poisson equation

$$\nabla^2\Phi = 4\pi G a^2\bar{\rho}_m\delta_m\ .$$

(a) Show that

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\bar{\rho}_m\delta_m = 0\ ,$$

where $H \equiv \dot{a}/a$.

(b) At early times ($a \ll a_{\rm eq}$), the universe is radiation dominated. Show that $u \equiv \delta_m/a$ then satisfies

$$\frac{d^2u}{da^2} + \frac{3}{a}\frac{du}{da} + \frac{1}{a^2}\left(1 - \frac{3}{2}\frac{a}{a_{\rm eq}}\right)u\ =\ 0\ .$$

[*Hint: You may use that during radiation domination* $\ddot{a} = -\dot{a}^2/a$.]

Determine the growing and decaying modes of δ_m in the radiation-dominated era. Justify any approximation you make.

(c) At late times ($a \gg a_{\rm eq}$), the universe is a mixture of dark matter and dark energy. The quantity $u \equiv \delta_m/H$ then satisfies [*you don't need to derive this!*]

$$\frac{d^2u}{da^2} + 3\frac{d\ln(aH)}{da}\frac{du}{da}\ =\ 0\ .$$

Show that the decaying mode is $\delta_m \propto H$, while the growing mode is

$$\delta_m \propto H\int_{a_i}^{a}\frac{\mathrm{d}\tilde{a}}{(\tilde{a}H)^3}\ ,$$

where a_i is the scale factor at some early time (but after $a_{\rm eq}$).

What are the growing and decaying modes of δ_m in the matter-dominated era? What is the asymptotic limit ($a \to +\infty$) of the growing mode solution in the dark energy-dominated era?

4

Slow-roll inflation is described by the following scalar field action

$$S = \int \mathrm{d}^4x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] ,$$

where ϕ is the inflaton field and $V(\phi)$ is its potential. Throughout this question you may ignore metric fluctuations and approximate the spacetime by the de Sitter line element, i.e. $\mathrm{d}s^2 = a^2(\tau) \left[\mathrm{d}\tau^2 - \mathrm{d}\boldsymbol{x}^2 \right]$, where $a(\tau) \approx -(H\tau)^{-1}$, with $3M_{\mathrm{pl}}^2 H^2 \approx V(\phi) \approx const.$

(a) Expand the scalar field into a background value and small fluctuations, i.e. $\phi(\tau, \boldsymbol{x}) = \bar{\phi}(\tau) + f(\tau, \boldsymbol{x})/a(\tau)$. Derive the quadratic action for f and show that the equation of motion for $f_{\boldsymbol{k}}(\tau) \equiv \int \frac{\mathrm{d}^3 x}{(2\pi)^{3/2}} \, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\tau, \boldsymbol{x})$ is

$$\ddot{f}_{\boldsymbol{k}} + \left(k^2 - \frac{\ddot{a}}{a} \right) f_{\boldsymbol{k}} = 0 ,$$

where the overdots denote derivatives with respect to conformal time τ.

[*Hint: You may freely drop boundary terms arising from integrations by parts. Moreover, you may assume that* $a^2 V'' \ll (aH)^2 = \frac{1}{2}\ddot{a}/a$.]

(b) Discuss qualitatively the quantisation of $f_{\boldsymbol{k}}$. In particular, write the associated quantum operator as

$$\hat{f}_{\boldsymbol{k}}(\tau) = u_k(\tau) \hat{a}_{\boldsymbol{k}} + u_k^*(\tau) \hat{a}^\dagger_{-\boldsymbol{k}} ,$$

state the meaning of $\hat{a}_{\boldsymbol{k}}$ and $\hat{a}^\dagger_{\boldsymbol{k}}$, cite the appropriate commutation relation and define the vacuum state $|0\rangle$. Verify that the appropriate solution for the mode function is given by

$$u_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau} \right) .$$

Why doesn't this solution include a piece proportional to $e^{+ik\tau}$?

(c) Compute the zero-point fluctuations,

$$\langle 0 | \delta\hat{\phi}^\dagger_{\boldsymbol{k}} \delta\hat{\phi}_{\boldsymbol{k}'} | 0 \rangle \equiv \frac{|u_k|^2}{a^2} \delta(\boldsymbol{k} - \boldsymbol{k}') ,$$

in the limit $k\tau \to 0$.

Briefly explain the relevance of this result for the primordial scalar and tensor fluctuations in the inflationary universe.

END OF PAPER