UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 11 June, 2013 9:00 am to 11:00 am

PAPER 47

CLASSICAL AND QUANTUM SOLITONS

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Throughout this paper we define *dimensionless* spacetime coordinates $x_0 = t$ and $x_1 = x$. The sine-Gordon field theory is defined to be the theory of a dimensionless real scalar field $\phi(x,t)$ with Lagrangian density,

$$\mathcal{L} \;=\; \frac{m^2}{\beta}\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi \,+\, (\cos\phi \,-\, 1)\right],$$

where m is a mass-scale and β is a dimensionless coupling. With these conventions the sine-Gordon equation reads,

$$\partial_\mu\partial^\mu\phi \,+\, \sin\phi \;=\; 0\,.$$

1

The *Bäcklund Tranform*, $\psi = \mathcal{B}_a[\phi]$ of a solution $\phi(x,t)$ of the sine-Gordon equation is defined by,

$$\begin{aligned}\frac{\partial}{\partial x_+}\left(\frac{\psi-\phi}{2}\right) \;&=\; a\,\sin\left(\frac{\psi+\phi}{2}\right)\\ \frac{\partial}{\partial x_-}\left(\frac{\psi+\phi}{2}\right) \;&=\; \frac{1}{a}\,\sin\left(\frac{\psi-\phi}{2}\right)\end{aligned}$$

where the parameter a is a real number, $x_\pm = (x \pm t)/2$ are lightcone coordinates in two-dimensional spacetime and ψ is defined up to an integration constant. Show that ψ is a solution of the sine-Gordon equation.

i) Let $\phi_0(x,t) = 0$ be the vacuum solution. Find the solution $\phi_a = \mathcal{B}_a[\phi_0]$ and explain its interpetation as a soliton. Find the velocity of the soliton as a function of a.

ii) Let $\phi_a = \mathcal{B}_a[\phi_0]$ as above and $\phi_b = \mathcal{B}_b[\phi_0]$ for some $b \neq a$. Now consider a third solution $\phi_{a,b}(x,t) = \mathcal{B}_a[\phi_b]$. You may assume without proof that the relation,

$$\tan\left(\frac{\phi_{a,b}-\phi_0}{4}\right) \;=\; \left(\frac{b+a}{b-a}\right)\tan\left(\frac{\phi_b-\phi_a}{4}\right)$$

holds for suitable values of the integration constants. Obtain the solution $\phi_{a,b}(x,t)$ in the special case $a = -1/b$. For convenience, you may also set the integration constants appearing in your expressions for ϕ_a and ϕ_b to zero. By considering the asymptotics of the resulting configuration at late and early times give an intepretation of the solution in terms of scattering. Find the topological charges and velocities of the scattered objects and determine the time delay in scattering.

2

Write an essay on quantization of solitons using the sine-Gordon model as an example. Your account should include a derivation of a general formula for the one-loop correction to the classical mass of a soliton. You should also give a qualitative discussion of the treatment of zero modes and of UV divergences.

3

An integrable two-dimensional relativistic field theory contains a charged particle A of mass m and its anti-particle $\bar{A}$. The two particle S-matrix of the theory is diagonal and the associated Faddeev–Zamolodchikov algebra is,

$$\begin{aligned} A(\theta_1)A(\theta_2) &= S_{AA}(\theta_1-\theta_2)A(\theta_2)A(\theta_1) \\ A(\theta_1)\bar{A}(\theta_2) &= S_{A\bar{A}}(\theta_1-\theta_2)\bar{A}(\theta_2)A(\theta_1) \end{aligned}$$

where θ_1 and θ_2 are particle rapidities. Define the properties of *unitarity* and *crossing symmetry* and explain how they constrain the S-matrix elements, $S_{AA}(\theta)$, $S_{A\bar{A}}(\theta)$.

Now suppose that the exact S-matrix element is given by the formula,

$$S_{AA}(\theta) = \frac{\sinh\left(\frac{\theta}{2}+i\frac{\lambda}{4}\right)}{\sinh\left(\frac{\theta}{2}-i\frac{\lambda}{4}\right)}$$

where λ is a coupling constant. Deduce the existence of a particle of charge $+2$ in the spectrum (where A has charge $+1$) and determine its mass.

Explain the *fusion* proceedure and use it to obtain the S-matrix element for the scattering of the new particle with A. Interpret the singularities of your result in terms of the mass spectrum of the theory.

END OF PAPER