UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 6 June, 2013 9:00 am 12:00 pm

PAPER 46

STRING THEORY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Write down the Nambu-Goto action for a closed string of tension T in a D-dimensional Minkowski spacetime with coordinates X^m $(m = 0, 1, \ldots, D-1)$. Explain why it is equivalent to the Polyakov action

$$S = -\frac{T}{2}\int d^2\sigma \sqrt{-\det\gamma}\,\gamma^{\mu\nu}\partial_\mu X^m \partial_\nu X^n \eta_{mn}\,,$$

where $\gamma_{\mu\nu}$ is an independent worldsheet metric in local coordinates σ^μ $(\mu = 0, 1)$. Use this equivalence to find the Nambu-Goto equation of motion for the worldsheet fields $X^m(t, \sigma)$.

Write down the generalization of the Polyakov action for a closed string in an arbitrary spacetime metric, and coupled to a 2-form potential b and scalar dilaton field Φ. Use your result to explain, briefly, how the vacuum value of the dilaton field appears in an expansion of the path-integral representation of the vacuum to vacuum amplitude in powers of the "string coupling constant" g_s.

The worldsheet of a closed Nambu-Goto string of tension T is embedded into a 5-dimensional Minkowski space such that $X^0 = t$ and

$$\begin{aligned} X^1 &= \frac{1}{2}\cos(\sigma - t)\,, \quad X^2 = \frac{1}{2}\sin(\sigma - t)\,, \\ X^3 &= \frac{1}{2}\cos(\sigma + t)\,, \quad X^4 = \frac{1}{2}\sin(\sigma + t)\,. \end{aligned} \tag{1}$$

What is the induced metric on the worldsheet? Use your result to verify that the Nambu-Goto equations are satisfied, and that the proper length L of the string is constant. By considering a rotating cartesian coordinate system for the Euclidean 4-space, show that the string is both circular and planar. What is the energy of the string? Why is it larger than TL?

The action for a closed Nambu-Goto string in Hamiltonian form is

$$S = \int dt \oint d\sigma \left\{\dot{X}^m P_m - \frac{1}{2}e\left[P^2 + (TX')^2\right] - uX'^m P_m\right\}.$$

What is the significance of the constraints imposed by the Lagrange multipliers e and u? What are the equations of motion for (X, P). Now consider an open Nambu-Goto string; what is the principle that determines the allowed boundary conditions at the ends of the string? Use this principle to show that it is consistent to impose free-end boundary conditions. Why must the endpoints of such a string move at the speed of light? Show also that it is consistent to impose boundary conditions that restrict the ends to move within a p-dimensional "plane". Explain, briefly, how the lowest energy states of a string with such boundary conditions are consistent with an interpretation of the p-plane as a planar p-brane.

2

Using the example of a mechanical system with phase-space action

$$S = \int dt \left\{ \dot{q}^I p_I - \lambda^i \varphi_i(q,p) \right\} , \qquad (I = 1, \ldots, N\,;\; i = 1, \ldots, n \leqslant N) \qquad (\star)$$

explain what is meant by the statement that the constraints $\varphi_i = 0$ imposed by the Lagrange multipliers λ^i are "first-class". Assuming that they *are* first-class, write down the gauge transformation of a function $F(q,p)$ on phase space generated by $\sum_i \xi^i \varphi_i$ for parameters ξ^i; what is the dimension of the physical phase space of this system?

The Fourier-space action for the open Nambu-Goto string in D spacetime dimensions with free ends is

$$S = \int dt \left\{ \dot{x}^m p_m + \sum_{k=1}^{\infty} \frac{i}{k} \alpha_{-k} \cdot \dot{\alpha}_k - \sum_{n \in \mathbb{Z}} \lambda_{-n} L_n \right\} , \quad L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} \alpha_k \cdot \alpha_{n-k} \, .$$

Explain briefly the origin and meaning of the canonical variables appearing in this action, and find their Poisson brackets by putting the action into the form $(\star)$. Compute the Poisson bracket algebra of the constraint functions L_n and hence show that the (classical) constraints are "first class". Find the gauge transformations of α_k, with parameters ξ_n, generated by $\sum_n \xi_{-n} L_n$.

By defining the light-cone components $(\alpha_k^{\pm}, \boldsymbol{\alpha}_k)$ of the D-vector α_k, use your result to show that the gauge invariance of the above string action may be partially fixed (assuming $\alpha_0^+ \neq 0$) by imposing the light-cone gauge conditions $\alpha_k^+ = 0$ for $k \neq 0$, and that the constraints may then be solved for α_k^- for $k \neq 0$. Write down the action in this gauge, and use it to find the canonical commutation relations of the operators $\boldsymbol{\alpha}_k$ of the quantum string in light-cone gauge. Show that physical states describe particles with squared masses given by the eigenvalues of an operator $\mathcal{M}^2 = (N - a)/\alpha'$, where you should define the level-number operator N and explain the meaning of the constants α' and a. Explain briefly why the first excited states must be massless and how this fact fixes a.

In the light-cone gauge quantization of the *closed* Nambu-Goto string in a $\text{Mink}_{D-1} \times S^1$ spacetime, the mass-squared operator is

$$\mathcal{M}^2 = \frac{2}{\alpha'} \left(N + \tilde{N} - 2 \right) + \left(\frac{n}{R} \right)^2 + \left(\frac{Rw}{\alpha'} \right)^2 ,$$

where R is the radius of the circle. Explain briefly the meaning of the operators $(N, \tilde{N})$, and the integers (n, w). Using the level-matching condition $N - \tilde{N} = wn$, find all massless states when $R = \sqrt{\alpha'}$.

3

A relativistic point particle of mass μ in D-dimensional Minkowski space has the action

$$S = \int dt \left\{ \dot{x}^m p_m - \frac{1}{2} e \left(p^2 + \mu^2\right) \right\} \qquad (m = 0, 1, \ldots, D-1).$$

Explain, with brief justification in the context of path-integrals, why the choice of gauge $e = \bar{e}$ (for constant $\bar{e}$) requires an addition to the classical gauge-fixed action of a term involving anticommuting Faddeev-Popov (anti)ghost variables (b, c). Write down the full gauge-fixed action $S[x, p; b, c]$ for $\bar{e} = 1$, and show that it is invariant under a BRST symmetry with anticommuting parameter Λ. What is the BRST charge, Q_{BRST}, and how does it generate the BRST transformations? Show that the BRST operator of the quantum theory satisfies $Q^2_{BRST} = 0$. Why is the physical state condition $Q_{BRST}|\text{phys}\rangle = 0$ equivalent to the Klein-Gordon equation?

In conformal gauge and with the corresponding FP ghost term, the Fourier-space action for the open Nambu-Goto string, in D spacetime dimensions, has commuting canonical variables α^m_k and anticommuting canonical variables (b_k, c_k). Write down the non-zero Poisson brackets of these variables, and the anticomutation relations of the corresponding operators of the quantum theory. In terms of these operators, the BRST operator is

$$Q_{BRST} = \frac{1}{2} \sum_{p \in \mathbb{Z}} \sum_{q \in \mathbb{Z}} c_{-p} \left[\alpha_q \cdot \alpha_{p-q} - (p-q) c_{-q} b_{p+q}\right] .$$

Let $L_m = \{b_m, Q_{BRST}\}$. Show that L_m, so defined, can be written as the sum of expressions $L^{(\alpha)}_m$ and $L^{(gh)}_m$ (which you should find) that are quadratic in, respectively, α_k and the (anti)ghost variables (b_k, c_k). Why is L_0 ambiguous? In what follows you may assume that this ambiguity has been eliminated by requiring the oscillator vacuum $|0\rangle$ to be annihilated by both Q_{BRST} and b_0.

By considering $[\{b_m, Q_{BRST}\}, L_n]$ show that

$$Q^2_{BRST} = 0 \quad \Rightarrow \quad [L_m, L_n] = (m-n) L_{m+n} .$$

Now show (assuming that $Q^2_{BRST} = 0$) that $||L_{-1}|0\rangle||^2 = \alpha_0^2 - 2$. By relating the left hand side to the commutator $[L_1, L_{-1}]$, deduce that

$$\langle 0|L_0|0\rangle = \frac{1}{2}\alpha_0^2 - 1 .$$

Now compute $||L_{-2}|0\rangle||^2$ and use your result to deduce that $D = 26$. Finally, use the fact that b_0 and Q_{BRST} annihilate the oscillator vacuum to determine α_0^2 in the string ground state.

4

The massless spinning particle action in a D-dimensional Minkowski spacetime has the action

$$S = \int dt \left\{ \dot{x}^m p_m + \frac{i}{2}\zeta^m \dot{\zeta}^n \eta_{mn} - \frac{1}{2} e\, p^2 - i\chi p \cdot \zeta \right\} ,$$

where the canonical variables $\zeta^m(t)$ and the Lagrange multiplier χ are *anticommuting.* What are the equations of motion? Show how the manifest Lorentz invariance of this action leads to the conclusion that there is an antisymmetric-tensor Noether charge J^{mn} of the form

$$J^{mn} = x^m p^n - x^n p^m + S^{mn} ,$$

where you should determine the spin part (S^{mn}) and verify that the equations of motion imply that $\dot{J}^{mn} = 0$. Use this model to explain how anticommuting variables can be used for a "classical" description of spin. In particular, write down the Poisson bracket relations for the anticommuting D-vector ζ and explain how Dirac quantization of this model yields a spinor wavefunction satisfying the massless Dirac equation.

Write down the Fourier space action for the open Neveu-Schwarz (NS) string (with free ends and in D spacetime dimensions) in terms of the centre-of-mass modes (x^m, p_m) and the D-vector Bose and Fermi oscillator variables, α_k and b_r respectively, where $k \in \mathbb{Z}$ and $r \in \mathbb{Z} + 1/2$. Your action should contain Lagrange multiplier terms imposing the constraints $L_n = 0$ and $G_r = 0$, where

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} \alpha_k \cdot \alpha_{n-k} + \frac{1}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}} r\, b_{n-r} \cdot b_r , \qquad G_r = \sum_{k\in\mathbb{Z}} \alpha_{-k} \cdot b_{r+k} .$$

How does the labelling of Fermi oscillators arise from boundary conditions for the open NS string? How is α_0 related to the centre of mass D-momentum? Write down the non-zero (anti)commutation relations obeyed by the oscillator variables in the "old-covariant" approach to quantization and state the defining properties of the oscillator vacuum $|0\rangle$. Assuming that L_0 is defined such that $(2L_0 - \alpha_0^2)|0\rangle = 0$, a state is said to be "physical" if, for some constant a,

$$(L_0 - a)\,|\text{phys}\rangle = 0 , \quad L_n|\text{phys}\rangle = 0 ,\ n > 0 ; \quad G_r|\text{phys}\rangle = 0 ,\ r > 0 .$$

Given that $|0\rangle$ is "physical", what kind of particle in the string spectrum does it describe? What is the condition for the state $A(p) \cdot b_{-\frac{1}{2}}|0\rangle$ of D-momentum p to be physical? Show that all physical states of this form have non-negative norm if and only if $a \leqslant 1/2$. What is special about the limiting $a = 1/2$ case?

END OF PAPER