UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Thursday, 6 June, 2013 1:30 pm to 4:30 pm

## PAPER 45

## THE STANDARD MODEL

*Attempt no more than* ***THREE*** *questions.*

*There are* ***FOUR*** *questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

Answer both (a) and (b) below.

(a) Consider a field theory with an $n$-component scalar field $\phi$ which interacts with itself through a potential $V(\phi)$. The potential is invariant under the following infinitesimal transformations of the field:

$$\phi \;\mapsto\; \phi + i\alpha^a t^a \phi\,,$$

where the $t^a$ are the generators of a group $G$ and the $\alpha^a$ are infinitesimal parameters. Take it as given that the potential is minimized by field configurations collectively denoted $\Phi_0 = \{\phi_0 | V(\phi_0) = V_{\text{min}}\}$ and that a given vacuum $\phi_0$ is invariant under transformations belonging to a subgroup $H$. That is, for any generator $\tilde{t}^i$ of $H$,

$$\tilde{t}^i \phi_0 \;=\; 0\,.$$

By expanding $V(\phi)$ about the vacuum $\phi_0$, prove that there are $\dim G - \dim H$ massless scalar modes.

(b) Let $\phi$ be a 3-component scalar field coupled to an $SU(2)$ gauge field with Lagrangian

$$\mathcal{L} \;=\; -\frac{1}{4}\text{Tr}\, F_{\mu\nu}F^{\mu\nu} \;+\; \frac{1}{2}D_\mu\phi \cdot D^\mu\phi \;-\; \frac{\lambda}{8}(|\phi|^2 - v^2)^2$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\epsilon^{abc}A^b_\mu A^c_\nu$ and $D_\mu = \partial_\mu + igt^a A^a_\mu$. It is useful to use the following matrix representation for the generators: $(t^a)_{jk} = -i\epsilon_{ajk}$. Show, at the classical level, that the vacuum spontaneously breaks $SU(2)$ to $U(1)$ and write down the Lagrangian in terms of physical fields. For convenience, you may begin by assuming the field fluctuates about the minimum as

$$\phi(x) \;=\; \begin{pmatrix} 0 \\ 0 \\ v + \eta(x) \end{pmatrix}.$$

Be sure to comment on the masses of the physical fields and briefly summarize the interactions felt by the fields.

Without doing any explicit calculation, briefly remark on what would have happened to the fields $\pi_1(x)$ and $\pi_2(x)$ if you had instead considered departures from the minimum of the type

$$\phi(x) \;=\; \begin{pmatrix} \pi_1(x) \\ \pi_2(x) \\ v + \eta(x) \end{pmatrix}.$$

Why is this model not suitable to describe the electroweak sector of the Standard Model, even after coupling the fields to fermions?

**2**

The following interaction terms contained in the Standard Model Lagrangian govern decays of the Higgs boson, $H$, to $W$ bosons and to $b$ quarks, respectively:

$$\begin{aligned}\mathcal{L}_{HWW} &= \frac{2m_W^2}{v}\, g^{\mu\nu} W_\mu^+ W_\nu^- H \\ \mathcal{L}_{H\bar{b}b} &= -\frac{m_b}{v}\bar{b}bH\,.\end{aligned}$$

($g^{\mu\nu}$ is the Minkowski metric.) Derive expressions for the partial decay rates $\Gamma(H \to W^+W^-)$ and $\Gamma(H \to \bar{b}b)$ which depend only on the particle masses, the Higgs VEV $v$, and numerical factors. Do not neglect any masses; do not consider hadronization effects.

Which decay is more dominant at large values of the Higgs boson mass?

[*Hints: Recall that the matrix element of a vector quantum field $W_\mu$ between the vacuum and a vector state $W(k, \varepsilon(k))$ is $\langle 0|W_\mu|W(k,\varepsilon(k))\rangle = \varepsilon_\mu(k)$. Also,*

$$\sum_{\text{spins}} \varepsilon_\mu^*(k)\varepsilon_\nu(k) \;=\; -g_{\mu\nu} + \frac{k_\mu k_\nu}{m_W^2}\,.$$

*In the Higgs boson's rest frame, the partial decay rate for its decay to 2 particles, say $X_1, X_2$, is given by*

$$\Gamma(H \to X_1 X_2) \;=\; \frac{1}{2m_H}\int \prod_{r=1}^{2}\left[\frac{d^3k_r}{(2\pi)^3 2k_r^0}\right](2\pi)^4\delta^{(4)}(p - k_1 - k_2)\sum_{\text{d.o.f.}}|\mathcal{M}|^2\,,$$

*where $p$ is the initial momentum and $k_1$, $k_2$ are the momenta of the final state particles. "d.o.f." stands for degrees-of-freedom.*]

**3**

Consider the process

$$e^+e^- \to \text{hadrons}$$

via annihilation of the electron and positron to a virtual photon. Find the cross section $\sigma$ in terms of the hadronic spectral density function $\rho_h(q^2)$, where $q$ is the sum of the electron and positron 4-momenta.

Explain why it is reasonable to expect that at high energies the leading-order cross section is given by $e^+e^- \to \bar{q}q$, where $\bar{q}q$ denotes a quark–antiquark pair. Calculate this contribution to $\sigma$. Clearly indicate the dependence on the number of colours and quark flavours. You may treat quarks with mass-squared less than $q^2$ as massless, and those with mass-squared greater than $q^2$ as infinitely massive, and therefore not produced. You may also treat $m_e$ as negligible.

[*Hint: The cross section for two spinless particles colliding to produce a generic final state $X$ is given by*

$$\frac{1}{|\vec{v}_1 - \vec{v}_2|} \frac{1}{4p_1^0 p_2^0} |\mathcal{M}_X|^2$$

*followed by integration over final state momenta and a sum over final state degrees-of-freedom. In the expression above $\vec{v}_j$ and $p_j$ are the 3-velocity and 4-momentum of the $j$-th initial state particle.*]

**4**

Write an essay on effective field theory. In particular, explain what we mean by an effective field theory, its realm of applicability, and how it is constructed. Describe a simple example, remarking how it illustrates the points made in your general discussion.

## END OF PAPER