UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 31 May, 2013 9:00 am to 12:00 pm

PAPER 44

ADVANCED QUANTUM FIELD THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The action for a *complex* $z(t)$, defined over time interval $0 \leqslant t \leqslant T$, is

$$S[z] = \int_0^T \mathrm{d}t\,(\dot{z}^* \dot{z} - \omega^2\, z^* z)\,, \qquad \dot{z}(t) = \frac{\mathrm{d}}{\mathrm{d}t} z(t)\,.$$

Show how to evaluate the functional integral over $z(t)$

$$K(z_f, z_i; T) = \int \mathrm{d}[z]\; e^{iS[z]}\,, \qquad z(0) = z_i\,,\; z(T) = z_f\,,$$

by expanding $z(t)$ about the classical solution $z_c(t)$, for which $S[z]$ is stationary at $z = z_c$ under independent variations of z and z^*, requiring $z_c(0) = z_i$, $z(T) = z_f$, to obtain

$$K(z_f, z_i; T) = \frac{1}{\det \Delta_\omega}\, e^{iS[z_c]}\,, \qquad \Delta_\omega = -\frac{\mathrm{d}}{\mathrm{d}t^2} - \omega^2\,,$$

$$S[z_c] = \frac{\omega}{\sin \omega T}\Big((|z_f|^2 + |z_i|^2) \cos \omega T - z_f^{\,*} z_i - z_i^{\,*} z_f \Big)\,.$$

Assume $\mathrm{d}[z]$ is defined so that $\det \Delta_0 = \pi i T$ and show that then $\det \Delta_\omega = \pi i\, \sin \omega T/\omega$.

Show that

$$\int \mathrm{d}^2 z\; K(\pm z, z; -iT) = \frac{e^{-\omega T}}{(1 \mp e^{-\omega T})^2} = \sum_{n=1}^{\infty} (\pm 1)^{n-1} n\, e^{-n\omega T}\,.$$

What is the interpretation of this result?

[*You may assume for integration over complex z*

$$\int \mathrm{d}^2 z\; e^{i\lambda z^* z} = \frac{\pi}{i\lambda} \quad \lambda \textit{ real},\; \lambda > 0\,, \quad \mathrm{d}^2 z = \mathrm{d}x \mathrm{d}y \;\; \textit{for } z = x + iy\,. \quad]$$

2

For the Lagrangian, in d-dimensions,

$$\mathcal{L}_F = -\tfrac{1}{2}\,\partial^\mu\phi\partial_\mu\phi - \tfrac{1}{2}\,m^2\phi^2 - \bar\psi(\gamma^\mu\partial_\mu + M)\psi\,,$$

where ϕ is a real scalar field and ψ a Dirac spinor field, define the two point correlation functions $\langle\phi(x)\phi(0)\rangle$ and $\langle\psi(x)\bar\psi(0)\rangle$ in terms of a functional integral. Derive the equations

$$(-\partial^2 + m^2)\langle\phi(x)\phi(0)\rangle = -i\delta^d(x)\,, \qquad (\gamma^\mu\partial_\mu + M)\langle\psi(x)\bar\psi(0)\rangle = -i\delta^d(x)\,,$$

and hence obtain the momentum space propagators for ϕ, ψ,

$$-i\,\frac{1}{p^2+m^2-i\epsilon}\,, \qquad -i\,\frac{-i\gamma^\mu p_\mu + M}{p^2+M^2-i\epsilon}\,.$$

Let $\hat\tau_n(p_1,\dots,p_n)$, $\sum_i p_i = 0$, be the amplitude corresponding to connected one particle irreducible graphs with n external ϕ lines after factoring off $i(2\pi)^d\delta^d(\sum_i p_i)$. For $\mathcal{L}_F$, $\hat\tau_2(p,-p) = -p^2 - m^2$. What is $\hat\tau_n$ for $n>2$? For an interaction

$$\mathcal{L}_I = -y\,\bar\psi\psi\,\phi\,,$$

show that there is a one loop contribution to $\hat\tau_2$ of the form

$$\hat\tau_2(p,-p)^{(1)} = -\frac{4y^2}{(2\pi)^d i}\int \mathrm{d}^d k\,\frac{M^2 - k\cdot(k-p)}{(k^2+M^2-i\epsilon)((k-p)^2+M^2-i\epsilon)}\,.$$

Using $k\cdot(k-p) = \frac{1}{2}(k^2+M^2) + \frac{1}{2}((k-p)^2+M^2) - \frac{1}{2}p^2 - M^2$ show that $\hat\tau_2(p,-p)^{(1)}$ has a pole as $\varepsilon = 4-d \to 0$ of the form

$$\hat\tau_2(p,-p)^{(1)} \sim -\frac{1}{\varepsilon}\,\frac{y^2}{16\pi^2}(a\,p^2 + b\,M^2)\,,$$

and determine a, b. Show how this divergence may be cancelled by adding counterterms $\mathcal{L}_{\text{c.t.}} = -\frac{1}{2}A\,\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}B\,\phi^2$ with $A \propto a, B\propto b$.

Show that in the bare Lagrangian $\phi \to \phi_0 = Z_\phi^{\frac{1}{2}}\phi$ and $m^2 \to m_0{}^2$ where

$$Z_\phi = 1 - \frac{4}{\varepsilon}\,\frac{y^2}{16\pi^2}\,.$$

Sketch the one loop graph which gives rise to a contribution to $\hat\tau_4{}^{(1)}$ and show, without any detailed calculation, that the resulting integral is divergent in 4 dimensions. Assume

$$\hat\tau_4{}^{(1)} \sim -\frac{8}{\varepsilon}\,\frac{y^4}{16\pi^2}\,.$$

How must $\mathcal{L} = \mathcal{L}_F + \mathcal{L}_I$ be modified if the one loop divergences are to be all cancelled by letting $\phi\to\phi_0$ and also by suitable redefinitions of the couplings in $\mathcal{L}$?

[*The gamma matrices are assumed to satisfy* $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}\,1$ *and also* $\mathrm{tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}$, $\mathrm{tr}(\gamma^\mu) = 0$. *The metric is* $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. *You may also use*

$$\frac{1}{(2\pi)^d i}\int \mathrm{d}^d k\,\frac{1}{k^2+m^2-i\epsilon} \sim -\frac{2}{\varepsilon}\,\frac{m^2}{16\pi^2}\,.$$]

3

Consider a renormalisable quantum field theory which has a single dimensionless coupling g and no mass parameters. Let $\langle \phi(x_1) \dots \phi(x_n) \rangle$ be the finite n-point correlation function for scalar fields ϕ determined by perturbation expansion of the quantum field theory as a series in g.

Explain how the perturbative result for $\langle \phi(x_1) \dots \phi(x_n) \rangle$ depends on a mass scale μ and obtain, with suitable assumptions, the RG equation

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + n\gamma(g) \right) \langle \phi(x_1) \dots \phi(x_n) \rangle = 0 \,. \qquad (*)$$

Assume the function C is defined by

$$\int \mathrm{d}^4 x \, e^{ip\cdot x} \langle \phi(x) \phi(0) \rangle = -i \, \frac{C(p^2/\mu^2, g)}{p^2} \,,$$

and obtain the solution of $(*)$ in the form

$$C(e^{2t} p^2/\mu^2, g) = f(t) \, C(p^2/\mu^2, g(t)) \,, \qquad (\dagger)$$

for suitable $g(t)$ and $f(t)$. How does this equation imply that μ is essentially arbitrary.

What is the behaviour of $C(p^2/\mu^2, g)$ for large p^2 if

(i) $\beta(g) > 0, \ 0 < g < g_*, \ \beta(g_*) = 0$,

(ii) for small g, $\beta(g) = -bg^3, \ \gamma(g) = cg^2, \ b > 0$? In this case show that $f(t) \propto t^{c/b}$ for large t.

Suppose that the theory depends on a mass m and $(*)$ is modified to

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + \delta(g) \, m \frac{\partial}{\partial m} + n\gamma(g) \right) \langle \phi(x_1) \dots \phi(x_n) \rangle = 0 \,.$$

Assuming now $C(p^2/\mu^2, m/\mu, g)$ show that $(\dagger)$ becomes

$$C(e^{2t} p^2/\mu^2, m/\mu, g) = f(t) \, C(p^2/\mu^2, e^{-t} m(t)/\mu, g(t)) \,,$$

for suitable $m(t)$. If $C(p^2/\mu^2, m/\mu, g)$ has a well defined limit as $m \to 0$ why does this imply that masses can be neglected at large energies if $\delta(g)$ is not too large?

4

For a theory with fields ϕ which has a continuous symmetry such that the action is invariant, $\delta_\epsilon S[\phi] = 0$, subject to a transformation $\delta_\epsilon \phi = \epsilon^a t_a \phi$ for ϵ^a infinitesimal show how, by letting $\epsilon^a \to \epsilon^a(x)$, to define an associated conserved current j_a^μ. How can j_a^μ be used to construct a conserved charge Q_a?

If for any $X(\phi)$ and

$$\langle X(\phi)\rangle = \int \mathrm{d}[\phi]\; X(\phi)\, e^{iS[\phi]}\,, \qquad \langle 1\rangle = 1\,,$$

show how to obtain, with appropriate assumptions, the Ward identity

$$-i\int \mathrm{d}^d x\; \epsilon^a(x)\frac{\partial}{\partial x^\mu}\langle j_a^\mu(x)\, X\rangle = \delta_\epsilon \langle X\rangle\,.$$

What is $-i\frac{\partial}{\partial x^\mu}\langle j_a^\mu(x)\,\phi(x_1)\dots\phi(x_n)\rangle$?

For a non abelian gauge theory with gauge field $A_{\mu a}(x)$ (a is a group index), anti-commuting ghost fields $c_a, \bar{c}_a$ and an auxiliary scalar field b_a the quantum action is determined by the Lagrangian

$$\mathcal{L}_q = -\tfrac{1}{4}\, F^{\mu\nu}\cdot F_{\mu\nu} + \partial^\mu b\cdot A_\mu + \tfrac{1}{2}\xi\, b\cdot b - \partial^\mu \bar{c}\cdot D_\mu c\,,$$

where $F_{\mu\nu a} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} + f_{abc}A_{\mu b}A_{\nu c}$, $(D_\mu c)_a = \partial_\mu c_a + f_{abc}A_{\mu b}c_c$, $X\cdot Y = X_a Y_a$ and f_{abc} is antisymmetric.

Show that $\mathcal{L}_q$ has a symmetry, with suitable assumptions, under

$$\delta_\theta(c_a,\bar{c}_a) = \theta(c_a, -\bar{c}_a)\,,\quad \delta_\epsilon(A_{\mu a}, c_a, \bar{c}_a, b_a) = \epsilon\big((D_\mu c)_a, -\tfrac{1}{2}f_{abc}c_b c_c, b_a, 0\big)\,,$$

for infinitesimal θ and anti-commuting ϵ. Verify that $\delta_{\epsilon'}\delta_\epsilon(A_{\mu a}, c_a, \bar{c}_a, b_a) = 0$.

[*You need to use* $\delta_\epsilon F_{\mu\nu a} = \epsilon\, f_{abc}F_{\mu\nu b}\, c_c$, $\delta_\epsilon D_\mu c = 0$ *but you also need to show why these results are true.*]

What are the corresponding conserved currents j_G^μ and j_B^μ?

In the quantum field theory there are associated conserved charges Q_G and Q_B where $Q_B^2 = 0$. Outline how Q_B can be used to construct the space of physical states of the theory. Why do we expect the physical states to be annihilated by Q_G?

END OF PAPER