UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 4 June, 2013 1:30 pm to 4:30 pm

PAPER 42

SYMMETRIES, FIELDS AND PARTICLES

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define the groups SU(2) and SO(3), and find their Lie algebras. Show that these Lie algebras, including their bracket structure, are isomorphic.

Define the group SU(3). Identify an SU(2) subgroup and an SO(3) subgroup of SU(3).

Consider the standard action of SU(3) on $\mathbb{C}^3$, and let

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

denote a general vector in $\mathbb{C}^3$. For the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, determine the orbit and the isotropy group.

The equation

$$z_1^2 + z_2^2 + z_3^2 = 1$$

defines a subset M of $\mathbb{C}^3$. Show that M is not an orbit of SU(3). What can you say about the action of the real SO(3) subgroup on M?

[*Hint: You may find it helpful to write* $\mathbf{z} = \mathbf{x} + i\mathbf{y}$, *where* $\mathbf{x}$ *and* $\mathbf{y}$ *are real vectors.*]

2

The gauge potential of a gauge theory in the (x, y) plane, with gauge group G, has components (A_x, A_y). Show that the field tensor has only one independent component, F_{xy}, and give the formula for this.

Let $g(x, y)$ be a G-valued function, and suppose

$$A_x = \alpha(\partial_x g)g^{-1} \quad , \quad A_y = \alpha(\partial_y g)g^{-1} \, ,$$

where α is a real constant. Evaluate F_{xy} in terms of g and its derivatives. Explain why, for certain values of α, F_{xy} vanishes for all $g(x, y)$.

Suppose now that G = SU(2) and that

$$g(x, y) = \exp\left(-\frac{1}{2}i(x\sigma_1 + y\sigma_2)\right)$$

where σ_a $(a = 1, 2, 3)$ are the Pauli matrices. Show that $g = \pm I$ at the origin and on an infinite number of circles in the plane. Calculate F_{xy} at the origin, and show that $F_{xy} = 0$ at all points on the circles.

3

What are meant by the *weights* of a representation of SU(3). Calculate, from the definitions, the weights of **3**, the fundamental representation of SU(3), and $\overline{\mathbf{3}}$, its complex conjugate.

Calculating with weights, show that

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1} \qquad (*)$$

and describe briefly what the representations **8** and **1** are, and why they are irreducible.

In the context of the quark model with approximate SU(3) flavour symmetry, discuss how the relation $(*)$ leads to a classification of meson states with spin/parity 0^-, including the pions. Briefly discuss the quark content and some physical properties of the meson states at the centre of the weight diagram.

4

Give an account of how the irreducible representations of

i) SO(4),

and

ii) SO(1,3)$^\uparrow$, the connected component of the Lorentz group,

may be constructed using irreducible representations of SU(2).

END OF PAPER